Capacitance and Dielectrics

**Capacitance**

General Definition:

\[ C = \frac{q}{V} \]

Special case for parallel plates:

\[ C = \frac{\varepsilon_0 A}{d} \]

**Potential Energy**

- I must do work to charge up a capacitor.
- This energy is stored in the form of electric potential energy.
- We showed that this is
  \[ U = \frac{Q^2}{2C} \]
- Then we saw that this energy is stored in the electric field, with a volume energy density
  \[ u = \frac{1}{2} \varepsilon_0 E^2 \]

**Potential difference and Electric field**

Since potential difference is work per unit charge,

\[ \Delta V = \int_a^b E \, dx \]

For the parallel-plate capacitor \( E \) is uniform, so

\[ V = Ed \]

Also for parallel-plate case Gauss’s Law gives

\[ E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} = Vd \]

so

\[ C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d} \]

**Spherical example**

A spherical capacitor has inner radius \( a = 3 \text{ mm} \), outer radius \( b = 6 \text{ mm} \). The charge on the inner sphere is \( q = 2 \text{ C} \). What is the potential difference?

From Gauss’s Law or the Shell Theorem, the field inside is

\[ E = \frac{kq}{r^2} \]

From definition of potential difference

\[ V = \int_a^b \frac{kq}{r} \, dr = kq \left[ \frac{1}{a} - \frac{1}{b} \right] \]

\[ = 9 \times 10^9 \times 2 \times 10^{-10} \left[ \frac{1}{3 \times 10^{-3}} - \frac{1}{6 \times 10^{-3}} \right] = 18 \times 10^{-10} \left( \frac{1}{3} - \frac{1}{6} \right) = 3 \times 10^3 \text{ V} \]

What is the capacitance?

\[ C = \frac{Q}{V} = \frac{(2 \text{ C})}{(3000 \text{ V})} = 6.7 \times 10^{-4} \text{ F} \]

A capacitor has capacitance \( C = 6 \mu \text{F} \) and charge \( Q = 2 \text{ nC} \). If the charge is increased to 4 nC what will be the new capacitance?

- (1) 3 \mu \text{F}
- (2) 6 \mu \text{F}
- (3) 12 \mu \text{F}
- (4) 24 \mu \text{F}
**Q. 25 - 1**

A capacitor has capacitance \( C = 6 \, \mu F \) and a charge \( Q = 2 \, nC \). If the charge is increased to 4 nC what will be the new capacitance?

**Solution:**

Capacitance depends on the structure of the capacitor, not on its charge.

(1) 3 \( \mu F \)  (2) 6 \( \mu F \)  (3) 12 \( \mu F \)  (4) 24 \( \mu F \)

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**Q. 25 - 2**

\( A = 0.1 \, m^2 \)

\( Q = 9 \times 10^{-9} \, C \)

\( E = ? \)

**Solution:**

Field at surface of conductor:

\[ E = \frac{\sigma}{\varepsilon_0} = \frac{9 \times 10^{-9}}{1 \times 10^{-12}} = 9 \times 10^3 \, V/m \]

1) \( 1 \times 10^6 \, V/m \)  (2) \( 900 \, V/m \)  (3) \( 9 \times 10^8 \, V/m \)

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**Capacitors in Parallel**

If several capacitors are connected in parallel as shown, we can consider the arrangement as a single capacitor.

By charge conservation:

\[ q_{tot} = q_1 + q_2 + \cdots \]

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**Capacitors in Series**

What if several capacitors are connected in series as shown?

Now charge conservation gives:

\[ q_{tot} = q_1 = q_2 = \cdots \]

And now it is the voltages that add:

\[ V_{tot} = V_1 + V_2 + \cdots \]
So now the voltage sum, when written in terms of the capacitances, gives the result for the series case:

\[
V_{\text{tot}} = V_1 + V_2 + \cdots
\]

\[
\frac{q_{\text{tot}}}{C_{\text{tot}}} = \frac{q_1}{C_1} + \frac{q_2}{C_2} + \cdots
\]

\[
\frac{1}{C_{\text{tot}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots
\]

Results for Capacitor Combinations

- Capacitors in parallel:
  \[
  C_{\text{tot}} = C_1 + C_2 + \cdots
  \]
- Capacitors in series:
  \[
  \frac{1}{C_{\text{tot}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots
  \]

Example: Problem 25-8

What is effective capacitance of the device?

Dielectric Materials

Effect of placing a dielectric between the plates of a charged capacitor:

- Applied electric field polarizes material
- This produces an induced surface charge
- This reduces field within material
- This reduces potential difference
- This increases capacitance

<table>
<thead>
<tr>
<th>Capacitance</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1)</td>
<td>10 (\mu F)</td>
</tr>
<tr>
<td>(C_2)</td>
<td>5.0 (\mu F)</td>
</tr>
<tr>
<td>(C_3)</td>
<td>4.0 (\mu F)</td>
</tr>
</tbody>
</table>

\[
1/C_{12} = 1/C_1 + 1/C_2 = 1/10 + 1/5 = 3/10
\]

\[
C_{12} = 10/3 = 3.3 \mu F
\]

\[
C_{\text{tot}} = C_{12} + C_3 = 3.3 + 4 = 7.3 \mu F
\]
Dielectric materials
Dielectric materials polarize under external electric field $E_0$
Polarization creates the opposing electric field: $-E'$
Total field $E = E_0 - E' / \kappa < E_0$
$\kappa$ is called dielectric constant
It shows how the field in a dielectric is weaker than in vacuum

Dielectric Constant
- So the effect of a dielectric is to decrease the field and increase the capacitance.
- The factor by which $E$ is decreased and $C$ is increased is defined to be the dielectric constant $\kappa$.

$$E = E_0 / \kappa$$
$$C = \kappa C_0$$

Example: Problem 25-48
Given a parallel-plate capacitor, dielectric-filled, with area $A = 100 \text{ cm}^2$, charge $Q = 890 \text{ nC}$, and electric field $E = 1.4 \text{ kV/mm}$.

(a) Find dielectric constant of the material.
(b) Find the induced charge.

(a) If there were no dielectric the field between the plates would be given by Gauss’s Law as

$$E_0 = \frac{(q / A)}{\varepsilon_0}$$

$$E_0 = \frac{8.9 \times 10^{-7} / 10^{-2}}{8.85 \times 10^{-12}} = 1.01 \times 10^7 \text{ V/m}$$

But by definition: $E = E_0 / \kappa$

So $\kappa = E_0 / E = 1.01 \times 10^7 / 1.4 \times 10^6 = 7.2$

(b) Gauss \Rightarrow

$$E = \frac{q_{\text{tot}} / A}{\varepsilon_0}$$

So $q_{\text{tot}} = \varepsilon_0 EA$

$= 8.85 \times 10^{-12} \times 1.4 \times 10^4 \times 10^{-2} = 12.4 \times 10^{-8} \text{ C} = 124 \text{ nC}$

But $q_{\text{tot}} = q - q'$

so $q' = q - q_{\text{tot}} = 890 \text{ nC} - 124 \text{ nC} = 766 \text{ nC}$