Exercises

1. (a) A disk with radius $R = 0.25 \text{ m}$ has a surface charge density of $\sigma = 0.5 \mu\text{C/m}^2$. What is the total charge of the disk?
(b) A ring with radius $R = 0.25 \text{ m}$ has a linear charge density of $\lambda = 0.5 \mu\text{C/m}$. What is the total charge of the ring?

(a) The surface charge density is constant, so the total charge is $Q = \sigma A$, where $A = \pi R^2$ is the area of the disk. Thus
$$Q = \sigma \pi R^2 = (0.5 \mu\text{C/m}^2) \pi (0.25 \text{ m})^2 = 0.098 \mu\text{C} \quad \text{or} \quad 9.8 \times 10^{-8} \text{C}$$

(b) The total charge of the ring is $Q = \lambda L$, where $L = 2\pi R$ is the length of the ring (i.e. its circumference). Thus
$$Q = \lambda (2\pi R) = (0.5 \mu\text{C/m}^2) 2\pi (0.25 \text{ m}) = 0.78 \mu\text{C} \quad \text{or} \quad 7.8 \times 10^{-7} \text{C}$$

2. (a) Six point charges, $q = 1 \mu\text{C}$ each, are arranged evenly on a circle with radius $r = 1 \text{ cm}$. What is the magnitude and direction of the electric field in the center of the circle? (Hint: think smarter not harder)
(b) What if five point charges were distributed on the same circle?

(a) The electric field must lie in the plane of the page. The system of charges has 6-fold rotational symmetry around the center of the circle; therefore, if the electric field at the center pointed in any particular direction, and we rotated the charges by $60^\circ$, it would point in some other direction, even though the charge distribution is the same. Thus, the electric field at the center of the circle must be [zero].

(b) This system has a similar rotational symmetry, so if the electric field did point in some direction, then rotating the whole system by $72^\circ$ will result in the same charge distribution, but a different electric field. This isn't possible, so the electric field must be [zero].
3. We want to find the electric field a distance \( h \) above the end of a finite line of charge, charge density \( \lambda \) and length \( L \), using integration. We can write the electric field as the integral

\[
\vec{E} = \int k \frac{dq}{\vec{R}^2} \vec{R}
\]

(a) Write an expression for \( dq \).
(b) Write an expression for \( \vec{R} \), as a function of \( x \) (the position on the line).
(c) What are the limits on the integral?
(d) Solve the integral and find \( \vec{E} \).
(e) Find the approximate electric field in the limit where \( L \gg h \). Is this the same electric field we found for an infinite line in class?

\[ \vec{R} = -x\hat{x} + h\hat{y} \]
and so the electric field due to that segment is

\[
d\vec{E} = k \frac{\lambda dx}{(x^2 + h^2)^{3/2}} (-x\hat{x} + h\hat{y})
\]
and so the electric field is

\[
\vec{E} = \int_0^L k \frac{\lambda dx}{(x^2 + h^2)^{3/2}} (-x\hat{x} + h\hat{y})
\]

\[
= \int_0^L -k\lambda \hat{x} \frac{dx}{(x^2 + h^2)^{3/2}} + \int_0^L k\lambda h \hat{y} \frac{dx}{(x^2 + h^2)^{3/2}}
\]

\[
= -k\lambda \hat{x} \int_0^L \frac{x dx}{(x^2 + h^2)^{3/2}} + k\lambda h \hat{y} \int_0^L \frac{dx}{(x^2 + h^2)^{3/2}}
\]

If we refer to the two integrals mentioned in class, we see that

\[
\vec{E} = -k\lambda \hat{x} \left[ -\frac{1}{\sqrt{x^2 + h^2}} \right]_0^L + k\lambda h \hat{y} \left[ \frac{x}{h^2\sqrt{x^2 + h^2}} \right]_0^L
\]

\[
= k\lambda \hat{x} \left[ \frac{1}{\sqrt{L^2 + h^2}} - \frac{1}{h} \right] + k\lambda h \hat{y} \left[ \frac{L}{h^2\sqrt{L^2 + h^2}} \right]
\]

\[
= k\lambda \frac{(h - \sqrt{L^2 + h^2})\hat{x} + L\hat{y}}{h\sqrt{L^2 + h^2}}
\]

(b) If \( L \gg h \), then \( \sqrt{L^2 + h^2} \approx L \) and \( h - \sqrt{L^2 + h^2} \approx h - L \approx -L \). Therefore

\[
\vec{E} \approx \frac{k\lambda(-L)\hat{x} + k\lambda L\hat{y}}{hL} = \frac{k\lambda}{h} (\hat{y} - \hat{x})
\]
Note that (i) the electric field dies off as \( 1/h \), as it does for a fully infinite line, but (ii) the electric field points up and to the left, and not just straight up.
If we think of a semi-infinite line which goes off to the left instead of the right, then its electric field would have the same magnitude, but point upwards and to the right. The total electric field of both lines is then $(2k\lambda/h)\hat{y}$, which is the electric field of an infinite line (as it should be).

4. To calculate the area of a surface, we divide the surface into a bunch of little areas $dA$, and add them all up: $A = \int dA$. To actually do the integral, we need to write $dA$ in terms of some coordinate system. For example, for a two-dimensional Cartesian coordinate system, we would write $dA = dx\, dy$. In polar coordinates, on the other hand, $dA = r\, dr\, d\phi$, where $r$ is the distance from the origin and $\phi$ is the angle measured from the positive $x$ axis.

(a) Using an integral, find the area of a circle with radius $R$.

We can calculate the total charge on a surface by dividing the surface up into a bunch of little areas $dA$, finding the charge on each one (call it $dq$), and then add them all up: $Q = \int dq$. Now the charge in a given area is the surface charge density $\sigma$ times its area, so $dq = \sigma\, dA$.

(b) Suppose the surface charge density on a disk with radius $R$ is $\sigma(r) = Kr$: that is, the charge density increases as one moves closer to the edge. Find an expression for the total charge on the disk.

(a) The area of a circle can be calculated by adding up all the little areas $dA$, like so: $A = \int dA$. In polar coordinates, $dA = r\, dr\, d\phi$, so

$$A = \int_0^{2\pi} \int_0^R r\, dr\, d\phi$$
$$= \int_0^{2\pi} \left[ \int_0^R r\, dr\right]\, d\phi$$
$$= \left[ \int_0^R r\, dr\right]\left[ \int_0^{2\pi} d\phi\right]$$
$$= \left[ \frac{1}{2} r^2 \right]_0^R [2\pi]$$
$$= \frac{1}{2} R^2 [2\pi] = \pi R^2$$

which is not a big surprise.

(b) Each little piece of the disk has charge

$$dq = \sigma\, dA = \sigma\, (r\, dr\, d\phi) = Kr^2\, dr\, d\phi$$

so the total charge is

$$Q = \int dq = \int_0^{2\pi} \int_0^R Kr^2\, dr\, d\phi$$
\[
= 2\pi \int_0^R Kr^2 \, dr \quad \text{Everything independent of } \phi, \text{ so that integral is easy}
\]
\[
= 2\pi \left[ \frac{1}{3} Kr^3 \right]_0^R
\]
\[
= \frac{2\pi}{3} KR^3
\]

Problems

\(\triangleright\) 5. Consider a thin ring, with uniform positive charge density \(\lambda\), sitting on the \(xy\) plane. Now consider a point inside the ring, and also on the \(xy\) plane.

(a) Using symmetry, explain why the electric field at this point cannot have a \(z\) component.

(b) Using symmetry, find the electric field at the center of the ring. Explain your reasoning.

\(\triangleright\) (a) The ring has reflection symmetry across the \(xy\)-plane. Any point on the \(xy\)-plane, when reflected by the \(xy\)-plane mirror, will end up back on top of itself. If the field at such a point had a \(z\) component—for example, if it pointed upward—then reflecting the entire system through the \(xy\) plane would result in the same ring and the same point, but a field pointing downward. That doesn’t preserve symmetry, so the field can’t have a \(z\) component. (You could also use rotational symmetry around the \(x\) or \(y\) axis.) This argument, by the way, works for any two-dimensional charge distribution: if a charge distribution lies on the \(xy\) plane, then the \(z\) component of the electric field at any point on the \(xy\) plane is zero.

(b) The field lies on the \(xy\) plane, so let’s suppose the field at the center of the ring points in some arbitrary direction. The ring has rotational symmetry around the center axis, so rotating the ring by any amount keeps the ring the same, but changes the field at that center point. That means that the field can’t point in any direction and maintain symmetry, and therefore the field at the center of the ring must be zero.

\(\triangleright\) 6. The figure shows four identical charges \(+q\) on the corners of a square. What’s wrong with the following argument? “Suppose the electric field at the star points downward. The square has 90° rotational symmetry, so when we rotate the square clockwise by 90°, the charges remain the same, but now the electric field points to the left. This is a contradiction, and so the electric field cannot point down at the star.”

\(\triangleright\) If I rotate the square by 90°, then the electric field points to the left, but the field points to the left at a different location, and so it isn’t a contradiction in this case. It would be a contradiction if we had put the star at the center of the square, however, because the star wouldn’t move.
7. Read the section entitled “Approximations” at the end of this homework, and then write approximations for the following expressions. All of your approximations should include a term which is proportional to $\epsilon$, which we assume to be small, but no term should include $\epsilon^2$, $\epsilon^3$, or any higher-order power of $\epsilon$. Zero is right out.

(a) $(1 + \epsilon)^{20}$
(b) $(9 - \epsilon)^{3/2}$
(c) $\frac{1}{\sqrt{2 + 5\epsilon}}$
(d) $\frac{1}{\sqrt{1 - 2\epsilon}} - \frac{1}{1 + 3\epsilon}$

In all four cases, we use the formula $(a + \epsilon)^n \approx a^n + na^{n-1}\epsilon$; note that the $\epsilon$ in this formula might be different from the $\epsilon$’s in the examples.

(a)

$(1 + \epsilon)^{20} \approx 1^{20} + 20(1^{19})\epsilon = 1 + 20\epsilon$

(This is a special case of the formula: $(1 + \epsilon)^n \approx 1 + n\epsilon$.)

(b)

$(9 - \epsilon)^{3/2} \approx 9^{3/2} + \frac{3}{2}9^{1/2}(-\epsilon) = 27 - \frac{3}{2}(3\epsilon) = 27 - \frac{9}{2}\epsilon$

(c)

$\frac{1}{\sqrt{2 + 5\epsilon}} = (2 + 5\epsilon)^{-1/2} \approx 2^{-1/2} - \frac{1}{2}(2^{-3/2})(5\epsilon) = \frac{1}{\sqrt{2}} - \frac{5\epsilon}{\sqrt{2}}$

(d) Both terms are of the form $(1 + \epsilon)^n$, and so we can write $(1 + \epsilon)^n \approx 1 + n\epsilon$:

$\frac{1}{\sqrt{1 - 2\epsilon}} - \frac{1}{1 + 3\epsilon} = (1 - 2\epsilon)^{-1/2} - (1 + 3\epsilon)^{-1}$

$\approx [1 + \left(-\frac{1}{2}\right)(-2\epsilon)] - [1 + (1)(3\epsilon)]$

$= 1 + \epsilon - [1 - 3\epsilon] = 4\epsilon$

Reading

Approximations: Sometimes, when one quantity in a formula is much larger than another, we would like to make an approximation. For example, if $A = 10^6$, and $\epsilon = 10^{-6}$, then $A + \epsilon \approx A$; $\epsilon$ is so small that it hardly matters. On the other hand, $A\epsilon = 1$, which in this case is not approximately $A$: approximations only
kick in when the small object and large object are added or subtracted from each other. The same thing is true with powers and multiples: for example, \(2A - 5\epsilon \approx 2A\), \(A^2 + \epsilon^2 \approx A^2\), etc.

One exception is when our approximation works out to be zero: for example, \(\frac{1}{a+\epsilon} - \frac{1}{a-\epsilon}\) where \(a \gg \epsilon\). (We will often use the Greek letter epsilon \(\epsilon\) to mean a small quantity.) Zero is never a useful approximation.

If removing \(\epsilon\) doesn’t work because it gives an approximation of zero, we need to try something else. One possibility is to add fractions together: for instance

\[
\frac{1}{a+\epsilon} - \frac{1}{a-\epsilon} = \frac{(a-\epsilon) - (a + \epsilon)}{(a+\epsilon)(a-\epsilon)} = \frac{-2\epsilon}{a^2 - \epsilon^2} \approx -\frac{2\epsilon}{a^2}
\]

Another method we can use is to use the approximation \((a + \epsilon)^n \approx a^n + na^{n-1}\epsilon\), where the exponent \(n\) can be positive or negative, integer or not. Some examples:

\[
\begin{align*}
(1 + \epsilon)^3 & \approx 1^3 + 3(1^2)\epsilon = 1 + 3\epsilon \\
\frac{1}{2 - \epsilon} & = (2 - \epsilon)^{-1} \approx 2^{-1} + (-1)(2^{-2})(\epsilon) = \frac{1}{2} + \frac{1}{4}\epsilon \\
\sqrt{4 + 3\epsilon} & = (4 + 3\epsilon)^{1/2} \approx 4^{1/2} + \frac{1}{2}(4^{-1/2})(3\epsilon) = 2 + \frac{3}{4}\epsilon
\end{align*}
\]

This is a rather imprecise introduction to approximations, but it gives you a sense of what I will be doing in some future problems and lectures.