Exercises

1. The figure shows an electron at the origin, and a grid marked off in nanometers.
(a) what is the electric field (in component form) at the point (1 nm, −2 nm), marked with a black star in the figure?
(b) If I put a proton at that point, what is the magnitude of the force that it feels?

(a) The electric field due to a point charge is given by the formula

\[ \vec{E} = k \frac{q_s \vec{r}}{r^2} \]

In this case, \( q_s = -e = -1.6 \times 10^{-19} \text{ C} \), so what we need to figure out is \( \vec{r} \). To get from the source to the target, we go in the +\( \hat{x} \) direction a distance of 1 nm, and in the −\( \hat{y} \) direction a distance of 2 nm, so \( \vec{r} = 1 \text{ nm}(\hat{x}) + 2 \text{ nm}(\hat{y}) \). The distance between source and target is then

\[ r = |\vec{r}| = \sqrt{(1 \text{ nm})^2 + (2 \text{ nm})^2} = (\sqrt{5}) \text{ nm} = 2.24 \times 10^{-9} \text{ m} \]

and

\[ \vec{r} = \frac{\vec{r}}{r} = \frac{10^{-9} \text{ m}(\hat{x}) + (2 \times 10^{-9} \text{ m})(\hat{y})}{2.24 \times 10^{-9} \text{ m}} = 0.446\hat{x} - 0.893\hat{y} \]

(Notice that the components of unit vectors have no dimension: not meters or anything, just pure numbers.)

And so the electric field at the star is

\[ \vec{E} = k \frac{q_s \vec{r}}{r^2} = \left(9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}\right) \frac{-1.6 \times 10^{-19} \text{ C}}{(2.24 \times 10^{-9} \text{ m})^2} (0.446\hat{x} - 0.893\hat{y}) \]

\[ = -2.87 \times 10^8 \text{ N}(0.446\hat{x} - 0.893\hat{y}) \]

\[ = 1.28 \times 10^8 \text{ N/C}(\hat{x}) + 2.56 \times 10^8 \text{ N/C}(\hat{y}) \]

(b) The force on a target charge \( q_t \) in an electric field is \( \vec{F} = q_t \vec{E} \), so the force on a proton (\( q_t = 1.6 \times 10^{-19} \text{ C} \)) is simply

\[ \vec{F} = (1.6 \times 10^{-19} \text{ C})(1.28 \times 10^8 \text{ N/C}(\hat{x}) + 2.56 \times 10^8 \text{ N/C}(\hat{y})) \]

\[ = 2.05 \times 10^{-11} \text{ N}(\hat{x}) + 4.10 \times 10^{-11} \text{ N}(\hat{y}) \text{ or } 20.5 \text{ pN}(\hat{x}) + 41.0 \text{ pN}(\hat{y}) \]
(A piconewton is $10^{-12}$ N.) We note that this force points to the left ($-\hat{x}$) and upwards ($\hat{y}$), which is back towards the electron: exactly what we expect for oppositely charged objects.

The question asks for the magnitude of the force, so we use the Pythagorean theorem:

$$|\vec{F}| = \sqrt{(20.5 \text{ pN})^2 + (41.0 \text{ pN})^2} = 45.8 \text{ pN}$$

or $4.58 \times 10^{-11}$ N.

\[ \Box \] 2. Two positive charges, one with charge $q_1 = 2 \mu C$ and one with charge $q_2 = 4 \mu C$, sit on the $y$ axis, 6 cm apart; the $x$ axis runs right between them. Find the electric field (magnitude and direction) on the $x$ axis, 4 cm to the right of the origin.

The electric field due to two charges is the sum of the electric field due to each charge:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = k \frac{q_1}{r_1} \vec{r}_1 + k \frac{q_2}{r_2} \vec{r}_2$$

where $\vec{r}_1$ is the vector from charge 1 to the target, and similarly from $\vec{r}_2$. We’re given $q_1$ and $q_2$. From the diagram, we see that

$$\vec{r}_1 = -(0.03 \text{ m})\hat{y} + (0.04 \text{ m})\hat{x} \quad \text{and} \quad \vec{r}_2 = +(0.03 \text{ m})\hat{y} + (0.04 \text{ m})\hat{x}$$

The length of both vectors is the same: $r_1 = r_2 = \sqrt{(0.03 \text{ m})^2 + (0.04 \text{ m})^2} = 0.05 \text{ m}$. Now we solve

$$\vec{E} = k \frac{q_1}{r_1} \vec{r}_1 + k \frac{q_2}{r_2} \vec{r}_2$$

$$= k \left[ \frac{2 \times 10^{-6} \text{ C}}{(5 \times 10^{-2} \text{ m})^3} \vec{r}_1 + \frac{4 \times 10^{-6} \text{ C}}{(5 \times 10^{-2} \text{ m})^3} \vec{r}_2 \right]$$

$$= (9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \frac{2 \times 10^{-6} \text{ C}}{125 \times 10^{-6} \text{ m}^3} [\vec{r}_1 + 2\vec{r}_2]$$

$$= 1.44 \times 10^8 \frac{\text{N}}{\text{C} \cdot \text{m}} \left[ -(0.03 \text{ m})\hat{y} + (0.04 \text{ m})\hat{x} \right] + [(0.06 \text{ m})\hat{y} + (0.08 \text{ m})\hat{x}]$$

$$= 1.44 \times 10^8 \frac{\text{N}}{\text{C} \cdot \text{m}} (0.03 \text{ m} \hat{y} + 0.12 \text{ m} \hat{x})$$

$$= \left[ 17.3 \text{ MN/C} \hat{x} + 4.3 \text{ MN/C} \hat{y} \right] = 1.73 \times 10^7 \text{ N/C} \hat{x} + 4.3 \times 10^6 \text{ N/C} \hat{y}$$

I asked for magnitude and direction, but that usually means I’m really looking for component form. However, the direction is largely to the right (since both charges are pushing to the right) and a little bit up (because the lower charge, being twice as big, is pushing harder). The magnitude is $1.8 \times 10^7 \text{ N/C}$ or 18 MN/C.
3. In the figure, the four particles form a square of edge length 

\[ a = 5.00 \text{ cm} \] and have charges \( q_1 = +10.0 \text{ nC}, \ q_2 = -20.0 \text{ nC}, \ q_3 = +20.0 \text{ nC}, \) and \( q_4 = -10.0 \text{ nC}. \) In unit-vector notation, what net electric field do the particles produce at the square’s center?

The electric field at the center of the square is the sum of the electric fields due to the four charges; and as is the case with Coulomb’s Law, the “tricky” part is to find the vector \( \vec{r} \) for each. For example, \( \vec{r}_1 \) is the vector from \( q_1 \) to the center, which can be gotten by moving a distance \( \frac{1}{2}a \) in the \( \hat{x} \) direction, and then \( \frac{1}{2}a \) in the \( -\hat{y} \) direction; thus \( \vec{r}_1 = \frac{1}{2}a\hat{x} - \frac{1}{2}a\hat{y} \). The length of this vector is 

\[ r_1 = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}a\right)^2} = \frac{1}{\sqrt{2}}a, \] 

and so its unit vector is 

\[ \hat{r}_1 = \frac{\vec{r}_1}{r_1} = \frac{1}{\frac{1}{\sqrt{2}}a} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y}). \]

By looking at the diagram, we see that the other three vectors all have the same length \( \frac{1}{\sqrt{2}}a \) (call this \( R \)) and their vectors are

\[ \begin{align*}
\vec{r}_2 &= \frac{1}{2}a(-\hat{x}) + \frac{1}{2}a(-\hat{y}) \quad \text{and so} \quad \hat{r}_2 = \frac{1}{\sqrt{2}}(-\hat{x} - \hat{y}) \\
\vec{r}_3 &= \frac{1}{2}a(-\hat{x}) + \frac{1}{2}a(\hat{y}) \quad \text{and so} \quad \hat{r}_3 = \frac{1}{\sqrt{2}}(-\hat{x} + \hat{y}) \\
\vec{r}_4 &= \frac{1}{2}a(\hat{x}) + \frac{1}{2}a(\hat{y}) \quad \text{and so} \quad \hat{r}_4 = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})
\end{align*} \]

Thus the electric field at the center is 

\[ \vec{E} = k \frac{q_1}{r_1^2} \hat{r}_1 + k \frac{q_2}{r_2^2} \hat{r}_2 + k \frac{q_3}{r_3^2} \hat{r}_3 + k \frac{q_4}{r_4^2} \hat{r}_4 
= k \frac{q_1}{(a/\sqrt{2})^2} \frac{1}{\sqrt{2}}(\hat{x} - \hat{y}) + k \frac{q_2}{(a/\sqrt{2})^2} \frac{1}{\sqrt{2}}(-\hat{x} - \hat{y}) + k \frac{q_3}{(a/\sqrt{2})^2} \frac{1}{\sqrt{2}}(-\hat{x} + \hat{y}) + k \frac{q_4}{(a/\sqrt{2})^2} \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}) 
= \frac{k}{(a^2/2)\sqrt{2}} [q_1(\hat{x} - \hat{y}) + q_2(-\hat{x} - \hat{y}) + q_3(-\hat{x} + \hat{y}) + q_4(\hat{x} + \hat{y})] 
= \frac{k\sqrt{2}}{a^2} [(q_1 - q_2 - q_3 + q_4)\hat{x} + (-q_1 - q_2 + q_3 + q_4)\hat{y}] 
\]

Now in this problem,

\[ q_1 - q_2 - q_3 + q_4 = [(10) - (-20) - (20) + (-10)] \text{ nC} = 0 \]

\[ -q_1 - q_2 + q_3 + q_4 = [-(10) - (-20) + (20) + (-10)] \text{ nC} = 20 \text{ nC} \]

Thus the electric field only has a \( y \) component: the \( x \) components of the fields here cancel. The field itself is 

\[ \vec{E} = \frac{\left(9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right) \sqrt{2}}{(0.05 \text{ m})^2} (20 \times 10^{-9} \text{ C}) \hat{y} = 1.02 \times 10^5 \text{ N/C} \hat{y} \] or \( 102 \text{kN/C} \)
Problems

Both of these problems require you to remember a bit from Physics 2130.

1. A uniform electric field (that is, \( \vec{E} \) is the same at every point) exists in a region between oppositely charged plates. An electron is released from rest at the surface of the negatively charged plate and strikes the surface of the opposite plate, 2.0 cm away, in a time \( 1.5 \times 10^{-8} \) s.

(a) What is the speed of the electron as it strikes the second plate?

(b) What is the magnitude of the electric field \( \vec{E} \)?

There is a uniform electric field between the plates, so the electron experiences a constant force, and thus undergoes constant acceleration. Therefore we can use all of those wonderful constant-acceleration formulae we learned in mechanics when dealing with gravity. Specifically,

\[
x = x_0 + v_0 t + \frac{1}{2} at^2 \quad \text{and} \quad v = v_0 + at
\]

Solving the second equation for \( a \) gives us \( a = (v - v_0)/t \); substituting into the first equation gives us

\[
x - x_0 = v_0 t + \frac{1}{2} (v - v_0) t = \frac{1}{2} (v + v_0) t
\]

which is a less familiar constant-acceleration formula (at least to me): the distance travelled is equal to the time travelled times the average speed, which in the case of constant acceleration is \( (v + v_0)/2 \). The initial velocity of the electron is \( v_0 = 0 \), so the speed of the electron as it strikes the second plate is

\[
v = 2 \frac{\Delta x}{t} = 2 \frac{0.02 \, \text{m}}{1.5 \times 10^{-8} \, \text{s}} = 2.7 \times 10^6 \, \text{m/s}
\]

which answers part a. For part b, we need the electric field, which means we need the force, which means we need the acceleration:

\[
E = \frac{F}{q} = \frac{ma}{q}
\]

The acceleration comes from the equation

\[
v = v_0 + at \quad \text{and so} \quad a = \frac{v}{t} = \frac{2.7 \times 10^6 \, \text{m/s}}{1.5 \times 10^{-8} \, \text{s}} = 1.8 \times 10^{14} \, \text{m/s}^2.
\]

The charge of an electron is \( q = 1.6 \times 10^{-19} \) C and the mass of an electron is \( m = 9.11 \times 10^{-31} \) kg, so

\[
E = \frac{(9.11 \times 10^{-31} \, \text{kg})(1.8 \times 10^{14} \, \text{m/s}^2)}{1.6 \times 10^{-19} \, \text{C}} = 1000 \, \text{N/C}.
\]

2. In the figure, a small, nonconducting ball of mass \( m = 1.0 \) mg (note the units) and charge \( q = 2.0 \times 10^{-8} \) C (distributed uniformly through its volume) hangs from an insulating thread that makes an angle \( \theta = 30^\circ \) with a vertical sheet which has a uniform charge density \( \sigma \) (shown in cross section); such a sheet creates a uniform electric field which points away from or towards the sheet (depending on the sign), with magnitude \( E = \frac{\sigma}{2\epsilon_0} \). Considering the gravitational force on the ball and assuming the sheet extends far vertically and into and out of the page, calculate the surface charge density \( \sigma \) of the sheet.

\[
E = \frac{\sigma}{2\epsilon_0} = \frac{1000 \, \text{N/C}}{2 \times 8.85 \times 10^{-12} \, \text{C}^2/\text{N} \cdot \text{m}^2} = 4.4 \times 10^{-10} \, \text{C/m}^2
\]
There are three forces on the ball, as shown in the figure to the right: the tension $T$ in the string, the weight $mg$ of the ball, and the electric force $qE$ due to the field of the plate. The tension force vector can be broken into its components $T \cos \theta$ pointing upward and $T \sin \theta$ pointing to the left. Assuming the ball is not moving, the net force on the ball is zero, and so

$$T \sin \theta = qE \quad \text{and} \quad T \cos \theta = mg$$

We use the second equation to find $T$:

$$T = \frac{mg}{\cos \theta} = \frac{(1.0 \times 10^{-6} \text{ kg})(9.8 \text{ m/s}^2)}{\cos 30^\circ} = \frac{9.8 \times 10^{-6} \text{ N}}{\sqrt{3}/2} = 1.13 \times 10^{-5} \text{ N}$$

Using this value, we solve for the electric field at the location of the ball:

$$E = \frac{T \sin \theta}{q} = \frac{(1.13 \times 10^{-5} \text{ N}) \sin 30^\circ}{2.0 \times 10^{-8} \text{ C}} = 283 \text{ N/C}$$

Now the electric field due to an infinite sheet of charge is $E = 2\pi k\sigma$. Therefore,

$$\sigma = \frac{1}{2\pi k} E = \frac{1}{2\pi(9 \times 10^9)}(283 \text{ N/C}) = \text{5.00 \times 10^{-9} C/m}^2 = 5.00 \text{ nC/m}^2.$$

\[\triangleleft\]