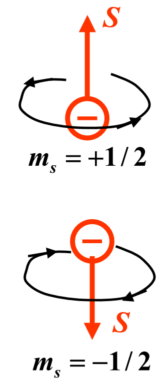
Fermions and Bosons

TODAY:

- Pauli exclusion principle
- Fermions and Bosons
- Lasers

Review: Electron spin

- Electrons have spin as an intrinsic quality. There are no non-spinning electrons.
- This gives an angular momentum vector $\vec{S}$
- There are only two possible quantum states for the electron spin: "up" and "down".



Spin and statistics

- We have seen how things go for one single electron moving in a potential well, or in a hydrogen atom.
- But for other atoms, and for solids, we have to deal with ***many electrons***, not just one.
- Two key ideas about states with several electrons:
  1. Electrons are ***indistinguishable.***
  2. Electrons are ***fermions.***

Statistics

Electrons are ***fermions,*** which means they obey Fermi-Dirac statistics, which in turn leads to the ***Pauli Exclusion Principle***: no two electrons can occupy the same quantum state.

All half-integer-spin particles (protons, neutrons, …) are fermions.

All integer-spin particles, such as photons, are ***bosons***, which means they obey Bose-Einstein statistics, and ***not*** the Pauli exclusion principle. In fact, photons ***prefer*** to have many in the same state.

Pauli Exclusion Principle

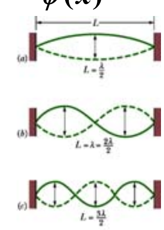
- No two electrons in the same state.
- That is, one electron for each set of quantum numbers: (n, l, m_l, m_s).
- This gives the ***periodic table!***
- Applies to spin-1/2 particles (fermions) such as electrons, protons, neutrons.
- Does not apply to photons.

Example: Problem 40-21

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. (Assume the electrons do not interact with each other; do not neglect spin!)

First recall the ***single-particle energies***, found by analogy with the modes of a vibrating string.

$\psi(x)$



$$\lambda = \frac{2L}{n} \qquad p = \frac{h}{\lambda} = \frac{nh}{2L}$$

$$E_n = \frac{p^2}{2m} = \frac{n^2 h^2}{8mL^2} = n^2 E_0$$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. (Assume the electrons do not interact with each other; do not neglect spin!)

One electron: $E_n = n^2 E_0 \quad E_0 = \dfrac{h^2}{8mL^2} \quad n = 1,2,3,\cdots$

If all 7 e's were in the lowest (n=1) single-particle state, then the lowest possible energy for the *system* would be just $7E_0$.

But Pauli forbids that. Only *two electrons* can have energy E_0, one with spin up, and one with spin down.

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

One electron: $E_n = n^2 E_0 \quad E_0 = \dfrac{h^2}{8mL^2} \quad n = 1,2,3,\cdots$

Make a list of occupied states and energies:

n=1	2	$2E_1 = 2*1E_0 = 2E_0$
n=2	2	$2E_2 = 2*4E_0 = 8E_0$
n=3	2	$2E_3 = 2*9E_0 = 18E_0$
n=4	1	$E_4 = 1*16E_0 = 16E_0$

Total ground state energy: $44E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. $E_n = n^2 E_0$

To get an excited state move one electron upward:

n=1	2	$2E_1 = 2*1E_0 = 2E_0$
n=2	2	$2E_2 = 2*4E_0 = 8E_0$
n=3	1	$1E_3 = 1*9E_0 = 9E_0$
n=4	2	$2E_4 = 2*16E_0 = 32E_0$

Energy of state (2,2,1,2): $51E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. $E_n = n^2 E_0$

Or another possibility is:

n=1	2	$2E_1 = 2*1E_0 = 2E_0$
n=2	1	$1E_2 = 1*4E_0 = 4E_0$
n=3	2	$2E_3 = 2*9E_0 = 18E_0$
n=4	2	$2E_4 = 2*16E_0 = 32E_0$

Energy of state (2,1,2,2): $56E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. $E_n = n^2 E_0$

Or another possibility is:

n=1	1	$1E_1 = 1*1E_0 = 1E_0$
n=2	2	$2E_2 = 2*4E_0 = 8E_0$
n=3	2	$2E_3 = 2*9E_0 = 18E_0$
n=4	2	$2E_4 = 2*16E_0 = 32E_0$

Energy of state (1,2,2,2): $59E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. $E_n = n^2 E_0$

Or another possibility is:

n=1	2	$2E_1 = 2*1E_0 = 2E_0$
n=2	2	$2E_2 = 2*4E_0 = 8E_0$
n=3	2	$2E_3 = 2*9E_0 = 18E_0$
n=5	1	$1E_5 = 1*25E_0 = 25E_0$

Energy of state (2,2,2,0,1): $53E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

$$E_n = n^2 E_0$$

Or another possibility is:

n=1	2	$2E_1 = 2*1E_0 = 2E_0$
n=2	2	$2E_2 = 2*4E_0 = 8E_0$
n=3	1	$1E_3 = 1*9E_0 = 9E_0$
n=4	1	$1E_4 = 1*16E_0 = 16E_0$
n=5	1	$1E_5 = 1*25E_0 = 25E_0$

Energy of state (2,2,1,1,1): $60E_0$

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

$$E_n = n^2 E_0$$

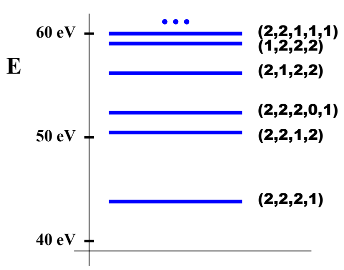
I think anything else will give a larger energy. (?)
To summarize we have found six energy levels for this seven-electron system.
There are an infinite number of energies, but these may be the six lowest.

(2,2,2,1): $44E_0$
(2,2,1,2): $51E_0$
(2,1,2,2): $56E_0$
(1,2,2,2): $59E_0$
(2,2,2,0,1): $53E_0$
(2,2,1,1,1): $60E_0$

Problem 40-21 (result)

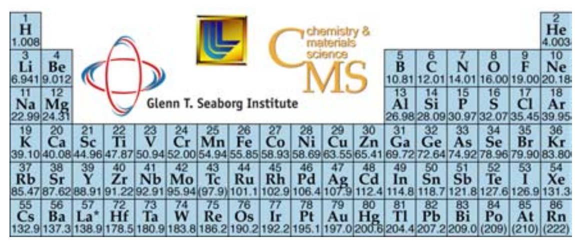
Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

Let's make an energy-level diagram for this system.
Suppose L = 0.62 nm so that $E_0 = 1$ eV.



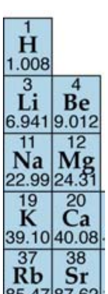
Now we can have quantum jumps with emission and absorption of photons!

Periodic Table



The first two columns

Alkalis and alkaline earths

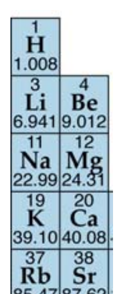


$$Na: \quad 1s^2\, 2s^2\, 2p^6\, 3s$$
$$Ca: \quad 1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6\, 4s^2$$

One or two valence electrons outside an inert closed-shell core.

The first two columns

Alkalis and alkaline earths

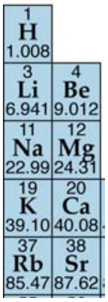


Closed-shell core of 10 electrons, like the inert-gas neon atom.

$$Na: \quad 1s^2\, 2s^2\, 2p^6\, 3s$$
$$Ca: \quad \underline{1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6}\, 4s^2$$

The first two columns

Alkalis and alkaline earths



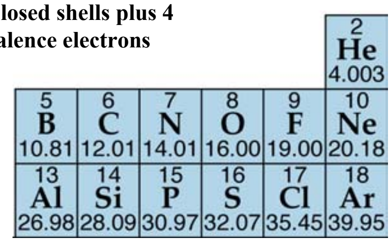
$$Na: \quad 1s^2\, 2s^2\, 2p^6\, 3s$$

$$Ca: \quad \underline{1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6}\, 4s^2$$

Closed-shell core of 18 electrons, like the inert-gas argon atom.

Carbon and silicon

Closed shells plus 4 valence electrons



$$C: \quad \underline{1s^2}\, 2s^2\, 2p^2 \qquad \text{He core}$$

$$Si: \quad \underline{1s^2\, 2s^2\, 2p^6}\, 3s^2\, 3p^2 \qquad \text{Ne core}$$

Example: Problem 40-28

Suppose two electrons in an atom have quantum numbers n=2 and l=1 (2p orbital). Keep in mind that the electrons have spin and are *indistinguishable*.

(Note this could be the two electrons in the 2p subshell of carbon, ground configuration $1s^2 2s^2 2p^2$.)

(a) How many states are possible for those two e's?

(b) If the Pauli principle did not apply, how many states would be possible?

Example continued

Counting the states for two electrons in the 2p subshell.

First: we see there are *6 single-particle states*: l=1 allows 3 values of m_l: (+1, 0, -1) and there can be 2 values of m_s: (+1/2, -1/2). Let us denote the 6 states using just the m_l, m_s values as:

(+ +), (+ −), (0 +), (0 −), (− +), (− −)

Next: Pretend electrons are *distinguishable:* Suppose we had one *blue electron* and one *red electron*. Think of these 6 states as 6 boxes and the 2 electrons as marbles. For each box in which we can put the blue marble, we can choose any one of 6 boxes to put the red marble in. So the total number of different states (permutations) is 6 × 6 = 36.

Example continued

Counting the states for two electrons in the 2p subshell.

6 single-particle states: $(m_l\ m_s)$ = (+ +), (+ −), (0 +), (0 −), (− +), (− −)

Number these 1,2,3,4,5,6.

But electrons are *indistinguishable:* the state (+ +)(+ −) (12) is physically exactly the same as the state (+ −)(+ +) (21). This means we change from permutations to combinations of the two marbles. (One marble in box 1 and one in box 2 is one combination no matter which marble is blue and which is red.)

B	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5	6
R	1	2	3	4	5	6	2	3	4	5	6	3	4	5	6	4	5	6	5	6	6

Counting: 6 + 5 + 4 + 3 + 2 +1

= 21 combinations   Answer to part (b)!

Example continued

Counting the states for two electrons in the 2p subshell.

6 single-particle states: $(m_l\ m_s)$ = (+ +), (+ −), (0 +), (0 −), (− +), (− −)

Number these 1,2,3,4,5,6.

Now we must require the Pauli Exclusion Principle. Cannot have both marbles in the same box.

B	~~1~~	1	1	1	1	1	~~2~~	2	2	2	2	~~3~~	3	3	3	~~4~~	4	4	~~5~~	5	~~6~~
R	~~1~~	2	3	4	5	6	~~2~~	3	4	5	6	~~3~~	4	5	6	~~4~~	5	6	~~5~~	6	~~6~~

Removes 6 of the 21, leaving 15 states for 2p2.

Answer to part (a)!

Recap: Identical particles

- All electrons are identical, all photons are identical, all protons are identical, etc.
- The wavefunction for a two-particle state could be written ψ(A,B), meaning particle #1 is at A and #2 is at B. But it could also be written ψ(B,A). The two states are physically the same.
- Therefore, $|\psi(A,B)|^2 = |\psi(B,A)|^2$.
- This gives two possibilities:
- ψ(A,B) = + ψ(B,A) or ψ(A,B) = + ψ(B,A)

Spin and Statistics

- The choice of statistics is determined by the spin of the particle.
- ψ(A,B) = + ψ(B,A): Bose-Einstein statistics (bosons)
- ψ(A,B) = − ψ(B,A): Fermi-Dirac statistics (fermions)
- It turns out that BE applies if spin is *integer*, and FD applies if spin is *half-integer*.
- Actually we mean

$$s = \frac{1}{2}\hbar \quad or \quad s = \frac{3}{2}\hbar \quad etc \quad for\ fermions$$

$$s = 0 \quad or \quad s = n\hbar \quad etc \quad for\ bosons$$

Why is this?

Textbooks either don't say or say it's too complicated to understand.

But in fact it's relatively simple:

"The Connection between Spin and Statistics", R.T.Deck and J.D.Walker (University of Toledo), Physica Scripta 63, 7-14, 2001.

One way to exchange two particles is to make a *rotation* – the result of which clearly depends on the spins.

Bosons and fermions

Fermi-Dirac statistics: electrons: *Atoms*

Two particles in the same state is forbidden.

Bose-Einstein statistics: photons: *Lasers*

Two particles in the same state is encouraged.

Stimulated Emission

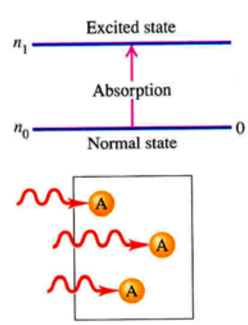
Light
Amplification by
Stimulated
Emission of
Radiation

Incident photon causes emission of a second photon into *exactly the same quantum state* as the original photon.

Exactly the opposite of the Pauli Principle.

Happens because photons are *bosons*, not fermions!

Absorption



(a)

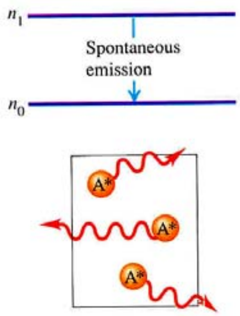
Photon of energy E_γ is ***absorbed***. Absorbing atom makes an ***upward*** quantum jump. Cannot happen unless energy is conserved:

$$E_\gamma = hf = E_1 - E_0$$

Light with $\lambda = hc / E_\gamma$

is ***removed*** from the incident beam, giving a ***dark*** spectrum line.

Spontaneous Emission



An ***excited*** atom makes a ***downward*** quantum jump. Photon of energy E_γ is emitted, with

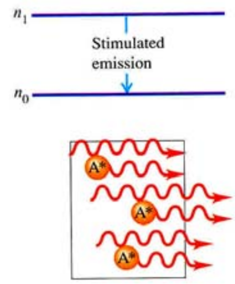
$$E_\gamma = hf = E_1 - E_0$$

So light of wavelength

$$\lambda = hc / E_\gamma$$

is ***emitted*** by hot gas, giving a ***bright*** spectrum line.

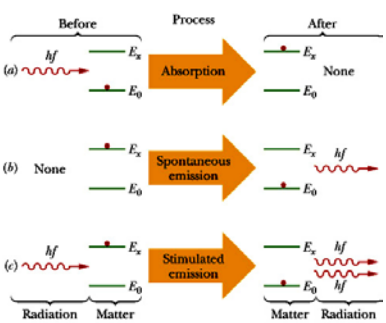
Stimulated Emission



Atom emits a new photon into *exactly the same quantum state* as the original photon. This keeps happening until there is a strong beam of many photons *all in the same quantum state.*

Produce intense, strongly collimated laser beam.

Recap: Photon-Atom Processes



Note if hf = E_x-E_0 we can have *both* absorption and stimulated emission. One will dominate depending on the *populations* of the two levels.

In thermal equilibrium more atoms are in the lower, so absorption dominates. No laser.

Must have *population inversion* to have laser.