Fermions and Bosons

TODAY:
- Pauli exclusion principle
- Fermions and Bosons
- Lasers

Review: Electron spin
- Electrons have spin as an intrinsic quality. There are no non-spinning electrons.
- This gives an angular momentum vector $S$
- There are only two possible quantum states for the electron spin: “up” and “down”.

Spin and statistics
- We have seen how things go for one single electron moving in a potential well, or in a hydrogen atom.
- But for other atoms, and for solids, we have to deal with many electrons, not just one.
- Two key ideas about states with several electrons:
  1. Electrons are indistinguishable.
  2. Electrons are fermions.

Statistics
Electrons are fermions, which means they obey Fermi-Dirac statistics, which in turn leads to the Pauli Exclusion Principle: no two electrons can occupy the same quantum state.
All half-integer-spin particles (protons, neutrons, …) are fermions.
All integer-spin particles, such as photons, are bosons, which means they obey Bose-Einstein statistics, and not the Pauli exclusion principle. In fact, photons prefer to have many in the same state.

Pauli Exclusion Principle
- No two electrons in the same state.
- That is, one electron for each set of quantum numbers: $(n, l, m_s, m_i)$.
- This gives the periodic table!
- Applies to spin-1/2 particles (fermions) such as electrons, protons, neutrons.
- Does not apply to photons.

Example: Problem 40-21
Seven electrons are trapped in a one-dimensional infinite potential well of width $L$. Find the lowest four energy levels of this system. (Assume the electrons do not interact with each other; do not neglect spin!)

First recall the single-particle energies, found by analogy with the modes of a vibrating string.

\[
\lambda = \frac{2L}{n} \quad p = \frac{\hbar}{\lambda} = \frac{n\hbar}{2L} \\
E_n = \frac{p^2}{2m} = \frac{n^2\hbar^2}{8mL^2} = n^2E_0
\]
Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. (Assume the electrons do not interact with each other; do not neglect spin!)

One electron: \( E_n = n^2 E_0 \), \( E_0 = \frac{\hbar^2}{8mL^2} \) \( n = 1, 2, 3, \cdots \)

If all 7 e’s were in the lowest (\( n=1 \)) single-particle state, then the lowest possible energy for the system would be just \( 7E_0 \).

But Pauli forbids that. Only two electrons can have energy \( E_0 \), one with spin up, and one with spin down.

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

To get an excited state move one electron upward:

\[
\begin{align*}
\text{n=1} & & 2E_4 = 2 \times 16E_0 = 32E_0 \\
\text{n=2} & & 2E_3 = 2 \times 9E_0 = 18E_0 \\
\text{n=3} & & 2E_2 = 2 \times 4E_0 = 8E_0 \\
\text{n=4} & & 2E_1 = 2 \times 1E_0 = 2E_0
\end{align*}
\]

Energy of state \( (2,2,1,2) \): 51\( E_0 \)

Or another possibility is:

\[
\begin{align*}
\text{n=1} & & 1E_5 = 1 \times 25E_0 = 25E_0 \\
\text{n=2} & & 2E_3 = 2 \times 9E_0 = 18E_0 \\
\text{n=3} & & 2E_2 = 2 \times 4E_0 = 8E_0 \\
\text{n=4} & & 2E_1 = 2 \times 1E_0 = 2E_0
\end{align*}
\]

Energy of state \( (2,2,2,2) \): 56\( E_0 \)

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

Or another possibility is:

\[
\begin{align*}
\text{n=1} & & 2E_5 = 2 \times 16E_0 = 32E_0 \\
\text{n=2} & & 2E_3 = 2 \times 9E_0 = 18E_0 \\
\text{n=3} & & 2E_2 = 2 \times 4E_0 = 8E_0 \\
\text{n=4} & & 1E_1 = 1 \times 1E_0 = 1E_0
\end{align*}
\]

Energy of state \( (1,2,2) \): 59\( E_0 \)

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system.

Or another possibility is:

\[
\begin{align*}
\text{n=1} & & 2E_4 = 2 \times 16E_0 = 32E_0 \\
\text{n=2} & & 2E_3 = 2 \times 9E_0 = 18E_0 \\
\text{n=3} & & 2E_2 = 2 \times 4E_0 = 8E_0 \\
\text{n=5} & & 1E_5 = 1 \times 25E_0 = 25E_0
\end{align*}
\]

Energy of state \( (2,2,0,1) \): 53\( E_0 \)
Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. \( E_n = n^2 E_0 \)

Or another possibility is:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2E_n )</th>
<th>( E_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 2E_2 = 2^2 E_0 = 4E_0 )</td>
<td>4E_0</td>
</tr>
<tr>
<td>3</td>
<td>11E_3 ( = 1^2 9E_0 = 9E_0 )</td>
<td>9E_0</td>
</tr>
<tr>
<td>4</td>
<td>( 1E_4 = 1^2 16E_0 = 16E_0 )</td>
<td>16E_0</td>
</tr>
<tr>
<td>5</td>
<td>( 1E_5 = 1^2 25E_0 = 25E_0 )</td>
<td>25E_0</td>
</tr>
</tbody>
</table>

Energy of state (2,2,1,1,1): 60\( E_0 \)

Problem 40-21 (continued)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. \( E_n = n^2 E_0 \)

I think anything else will give a larger energy. (?) To summarize we have found six energy levels for this seven-electron system. There are an infinite number of energies, but these may be the six lowest.

(2,2,1,1): 44\( E_0 \)
(2,2,1,2): 51\( E_0 \)
(2,1,2,2): 56\( E_0 \)
(1,2,2,2): 59\( E_0 \)
(2,2,2,0,1): 53\( E_0 \)
(2,2,1,1,1): 60\( E_0 \)

Problem 40-21 (result)

Seven electrons are trapped in a one-dimensional infinite potential well of width L. Find the lowest four energy levels of this system. Let’s make an energy-level diagram for this system. Suppose \( L = 0.62 \text{ nm} \) so that \( E_0 = 1 \text{ eV} \).

Periodic Table

The first two columns

**Alkalis and alkaline earths**

Na: \( 1s^2 2s^2 2p^6 3s \)

Ca: \( 1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 \)

One or two valence electrons outside an inert closed-shell core.

The first two columns

**Alkalis and alkaline earths**

Na: \( 1s^2 2s^2 2p^6 3s \)

Ca: \( 1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 \)

Closed-shell core of 10 electrons, like the inert-gas neon atom.
### The first two columns
Alkalis and alkaline earths

Na: \[ 1s^2 \ 2s^2 \ 2p^6 \ 3s \]
Ca: \[ 1s^2 \ 2s^2 \ 2p^6 \ 3s^2 \ 3p^6 \ 4s^2 \]

Closed-shell core of 18 electrons, like the inert-gas argon atom.

### Carbon and silicon
Closed shells plus 4 valence electrons

<table>
<thead>
<tr>
<th>( n )</th>
<th>( l )</th>
<th>( m_l )</th>
<th>( m_s )</th>
</tr>
</thead>
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<td>0</td>
<td>(+1/2)</td>
<td>(+1/2)</td>
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<tr>
<td>1</td>
<td>1</td>
<td>(+1)</td>
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</tr>
<tr>
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<td>2</td>
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<td>(+2)</td>
</tr>
<tr>
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<td>3</td>
<td>(+3)</td>
<td>(+3)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(+1/2)</td>
<td>(+1/2)</td>
</tr>
<tr>
<td>2</td>
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<td>(-3)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(+4)</td>
<td>(-4)</td>
</tr>
</tbody>
</table>

\( C: \ 1s^1 \ 2s^2 \ 2p^2 \)
\( Si: \ 1s^2 \ 2s^2 \ 2p^6 \ 3s^2 \ 3p^2 \)

### Example: Problem 40-28
Suppose two electrons in an atom have quantum numbers \( n=2 \) and \( l=1 \) (2p orbital). Keep in mind that the electrons have spin and are indistinguishable.

(Note this could be the two electrons in the 2p subshell of carbon, ground configuration 1s^2 2s^2 2p^2.)

(a) How many states are possible for those two e’s?
(b) If the Pauli principle did not apply, how many states would be possible?

### Example continued
Counting the states for two electrons in the 2p subshell.

First: we see there are 6 single-particle states: \( l=1 \) allows 3 values of \( m_l \): (+1, 0, -1) and there can be 2 values of \( m_s \): (+1/2, -1/2). Let us denote the 6 states using just the \( m_l, m_s \) values as:

- \(++\) \(+−\) 0+ 0− −+ −−

Number these 1, 2, 3, 4, 5, 6.

Counting: \[ 6 \] + 5 + 4 + 3 + 2 + 1 = 21 combinations Answer to part (b)!}

### Example continued
Next: Pretend electrons are distinguishable: Suppose we had one blue electron and one red electron. Think of these 6 states as 6 boxes and the 2 electrons as marbles. For each box in which we can put the blue marble, we can choose any one of 6 boxes to put the red marble in. So the total number of different states (permutations) is \( 6 \times 6 = 36 \).

Now we must require the Pauli Exclusion Principle. Cannot have both marbles in the same box.

Removes 6 of the 21, leaving 15 states for 2p^2.

Answer to part (a)!
Recap: Identical particles

- All electrons are identical, all photons are identical, all protons are identical, etc.
- The wavefunction for a two-particle state could be written $\psi(A,B)$, meaning particle #1 is at $A$ and #2 is at $B$. But it could also be written $\psi(B,A)$. The two states are physically the same.
- Therefore, $|\psi(A,B)|^2 = |\psi(B,A)|^2$.
- This gives two possibilities:
  - $\psi(A,B) = + \psi(B,A)$ or $\psi(A,B) = + \psi(B,A)$

Spin and Statistics

- The choice of statistics is determined by the spin of the particle.
  - $\psi(A,B) = + \psi(B,A)$: Bose-Einstein statistics (bosons)
  - $\psi(A,B) = - \psi(B,A)$: Fermi-Dirac statistics (fermions)
- It turns out that BE applies if spin is integer, and FD applies if spin is half-integer.
- Actually we mean $s = \frac{1}{2}$, $s = \frac{3}{2}$ etc for fermions
  - $s = 0$ or $s = n\hbar$ etc for bosons

Why is this?

Textbooks either don’t say or say it’s too complicated to understand.
But in fact it’s relatively simple:


One way to exchange two particles is to make a rotation – the result of which clearly depends on the spins.

Bosons and fermions

Fermi-Dirac statistics: electrons: *Atoms*

Two particles in the same state is forbidden.

Bose-Einstein statistics: photons: *Lasers*

Two particles in the same state is encouraged.

Stimulated Emission

Light Amplification by Stimulated Emission of Radiation

Incident photon causes emission of a second photon into exactly the same quantum state as the original photon.

Exactly the opposite of the Pauli Principle.

Happens because photons are *bosons*, not fermions!

Absorption

Photon of energy $E_\gamma$ is absorbed. Absorbing atom makes an upward quantum jump. Cannot happen unless energy is conserved:

$$E_\gamma = hf = E_1 - E_0$$

Light with $\lambda = hc / E_\gamma$ is removed from the incident beam, giving a dark spectrum line.
### Spontaneous Emission

An *excited* atom makes a **downward** quantum jump. Photon of energy $E_\gamma$ is emitted, with

$$E_\gamma = hf = E_1 - E_0$$

So light of wavelength

$$\lambda = \frac{hc}{E_\gamma}$$

is emitted by hot gas, giving a **bright** spectrum line.

### Stimulated Emission

Atom emits a new photon into **exactly the same quantum state** as the original photon. This keeps happening until there is a strong beam of many photons **all in the same quantum state**.

Produce intense, strongly collimated laser beam.

### Recap: Photon-Atom Processes

Note if $hf = E_1 - E_0$ we can have both absorption and stimulated emission. One will dominate depending on the population of the two levels.

In thermal equilibrium more atoms are in the lower, so absorption dominates. No laser.

Must have **population inversion** to have laser.