Relativity III

- **Today:**
  - Time dilation examples
  - The Lorentz Transformation
  - Four-dimensional spacetime
  - The invariant interval
  - Examples

Review: Kinetic Energy

General relation for *total* energy: $E = \gamma mc^2$

Rest energy, v=0: $E = mc^2$

Kinetic energy: $K = E - mc^2 = (\gamma - 1)mc^2$

Momentum: $p = \gamma mv$

Relation between momentum and energy: $E^2 = (mc^2)^2 + (pc)^2$

Exact vs non-relativistic calculations

Last time we saw that for the Stanford LINAC, a nonrelativistic calculation was terribly wrong (3 cm vs 2 miles).

That was a case where K >> mc^2. (ER case)

Now let's look at the case K << mc^2. (NR case)

Example: He beam from THIA

K = 300keV v = ?

$$mc^2 = 4 \times 1\ GeV = 4 \times 10^9\ eV \gg K$$

So non-relativistic calculation should be OK.

$$K = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2K}{m}}$$

$$\frac{v}{c} = \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{6 \times 10^5\ eV}{4 \times 10^9\ eV}} = \sqrt{1.5 \times 10^{-4}} = .01224745$$

$$v = .0122c = 3.67 \times 10^6\ m/s = 3.67\ mm/ns$$

Compare with exact relativistic answer

$$K = (\gamma - 1)mc^2$$

$$\gamma = 1 + \frac{K}{mc^2} = 1 + \frac{3 \times 10^5\ eV}{4 \times 10^9\ eV} = 1 + 7.5 \times 10^{-5}$$

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(1.000075)^2}} = \sqrt{1 - 0.99985} = .01224676$$

Compare with non-relativistic approximation:

$$\frac{1224745 - 1224676}{1224676} = 6 \times 10^{-5} = .006\ \%\ \text{error}$$

Q.37-3

An electron ($mc^2 = 0.5$ MeV) moves with a speed v = 0.94c so that $\gamma = 3$.

What is its kinetic energy?

1. 0.1 MeV
2. 0.5 MeV
3. 1.0 MeV
4. 2.0 MeV
5. 5.0 MeV

Q.37-3

$mc^2 = 0.5$ MeV, $\gamma = 3$: K = ?

$$K = E - mc^2 = (\gamma - 1)mc^2$$
$$= (3-1)\ 0.5\ MeV = 1.0\ MeV$$

1. 0.1 MeV
2. 0.5 MeV
3. 1.0 MeV
4. 2.0 MeV
5. 5.0 MeV

Time Dilation

A lab observer compares two stationary clocks against a clock moving with speed v, as it passes first one then the other. Lab clocks give Δt, moving clock Δt_0.

$$\Delta t = \gamma \Delta t_0 \geq \Delta t_0$$

Time measured in lab (Δt) is greater than proper time Δt_0 measured by co-moving observer.

"Moving clocks run slow".

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \geq 1$$

Example: Problem 37-21

A clock moves along the x axis at speed v = 0.6c and reads zero as it passes the origin. What time does the clock read as it passes x = 180 m?

Lab time: $\Delta t = \frac{x}{v} = \frac{180\ m}{.6 \times 3 \times 10^8\ m/s} = 1\ \mu s$

Gamma factor: $\gamma = \frac{1}{\sqrt{1-.6^2}} = \frac{1}{.8} = 1.25$

Proper time: $\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{1\ \mu s}{1.25} = 0.8\ \mu s$

Q.37-4

A clock moves relative to a laboratory, at speed v such that γ=5. During the time taken for the moving clock to advance 10 ns, how much time elapses according to the lab clocks?

(1) 0.5 ns (2) 2 ns (3) 5 ns (4) 10 ns (5) 50 ns

Q.37-4

A clock moves relative to a laboratory, at speed v such that γ=5. During the time taken for the moving clock to advance 10 ns, how much time elapses according to the lab clocks?

Solution:

$$\Delta t = \gamma \Delta t = 5 \times 10\ ns = 50\ ns$$

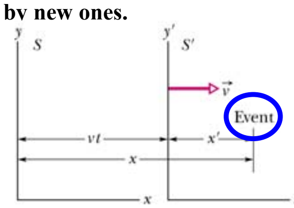
(1) 0.5 ns (2) 2 ns (3) 5 ns (4) 10 ns (5) 50 ns

Lorentz transformation

Einstein found that the old "self-evident" laws for transformations between inertial frames must be replaced by new ones.

Galileo: $x' = x - vt$

$t' = t$



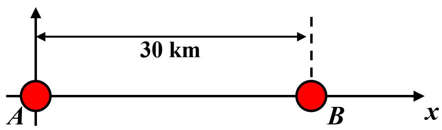
Einstein: use the *Lorentz transformation:*

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx/c^2)$$

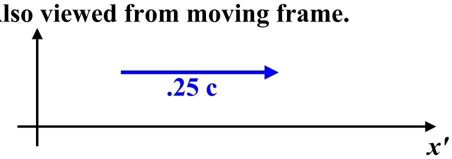
where $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

Example: Problem 37-19

Two flashbulbs triggered simultaneously.



Also viewed from moving frame.

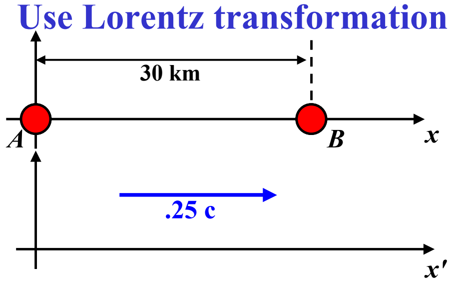


Use Lorentz transformation

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx/c^2)$$

For each event, knowing x,t, calculate x',t'.

Use Lorentz transformation



Coordinates in lab frame:

$$t_A = t_B = 0, \qquad x_A = 0, \qquad x_B = L = 30\ km$$

Coordinates in moving frame:

$$t'_A = 0, \qquad x'_A = 0, \qquad t'_B = ?$$

Example continued

Use Lorentz transformation equation for time of event B in moving frame.

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(.25)^2}} = 1.0328$$

$$t'_B = \gamma\left(t_B - vx_B/c^2\right) = -\gamma vL/c^2$$

$$= -\frac{1.0328 \times 0.25 \times 3 \times 10^4\ m}{3 \times 10^8\ m/s} = -25.8\ \mu s$$

But t'_A=0. So B happens before A as seen in moving frame!

Example summarized

Times in lab frame:

$$t_A = t_B = 0$$

Times in moving frame:

$$t'_A = 0 \qquad t'_B = -25.8\ \mu s$$

Events are not simultaneous in moving frame.

B is first, then A is 25.8 microseconds later.

The spacetime interval

- Our text doesn't stress this point, but there is another way of expressing laws of special relativity which is often simpler than using the Lorentz transformation equations or going back to the "two postulates".
- This is the *invariant spacetime interval.*

The spacetime continuum

Another way of expressing laws of special relativity which is often simpler than using the Lorentz transformation equations.

- **Instead of thinking of space and time separately, think of a *four-dimensional spacetime*. The "points" in this spacetime are really *events.***
- **Then the "distance" between events is called the *spacetime interval.***
- **Now relativity follows from the fundamental assumption that the *spacetime interval* is *invariant:* the same for all inertial observers.**

The spacetime interval

Given two ***events*** *(x_1,t_1)* and *(x_2,t_2)*. As seen by inertial observer ***O*** they are separated by intervals

$$\Delta x = x_2 - x_1$$
$$\Delta t = t_2 - t_1$$

As seen by another observer ***O'*** these intervals are different (***relative***).

$$\Delta x' \neq \Delta x$$
$$\Delta t' \neq \Delta t$$

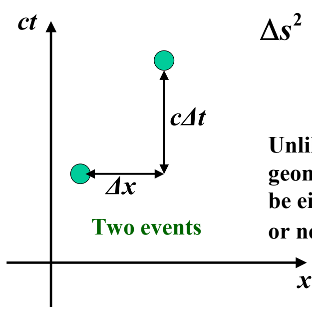
But the **space-time interval** $\Delta s^2 = c^2\Delta t^2 - \Delta x^2$ is the same (***invariant***):

$$\Delta s' = \Delta s$$

The spacetime diagram

An alternative to the Lorentz transformation equations is the invariant spacetime interval:

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2$$
$$\Delta s' = \Delta s$$



Unlike ordinary (Euclidean) geometry, this interval can be either positive (timelike) or negative (spacelike).

Example

Derive time dilation using Δs' = Δs.

Two events occur at the same place: $\Delta x = 0, \quad \Delta s = c\Delta t$

Same events seen by observer moving with speed v:

$$\Delta x' = v\Delta t', \quad (\Delta s')^2 = c^2(\Delta t')^2 - (\Delta x')^2$$
$$(\Delta s')^2 = c^2(\Delta t')^2 - v^2(\Delta t')^2$$
$$\Delta s' = c\Delta t'\sqrt{1-(v/c)^2}$$

So now use invariance of spacetime interval:

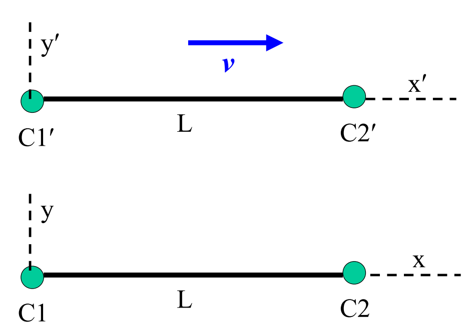
$$\Delta s' = \Delta s \text{ gives } \Delta t' = \Delta t / \sqrt{1-(v/c)^2} = \gamma\Delta t$$

All Observers are Equivalent

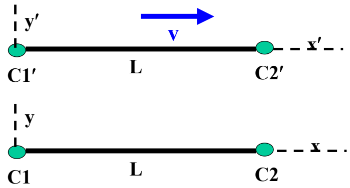
- **In a moving reference frame, lengths contract, time dilates. "Moving rods are short and moving clocks are slow!"**
- **How is this possible if all inertial observers are equivalent?**
- **If B moves with speed v relative to A, then B's clocks are slow as measured by A.**
- **But according to B, it's A that's moving, so A's clocks are slow as measured by B.**
- **Is this possible?**
- **Yes, because they also disagree about how the clocks are originally synchronized.**
- **A careful analysis shows that the effect is perfectly symmetrical: *"Each observer thinks the other is using slow clocks!"***

Simple Problem to Show Symmetry

Observers O and O' with relative speed v.



Use invariant spacetime interval

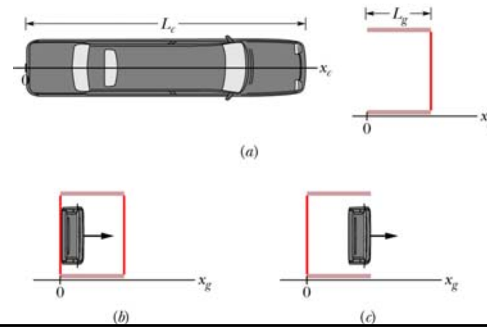


- **Four events:** **(a)** C1′ passes C1, **(b)** C1′ passes C2, **(c)** C2′ passes C1, **(d)** C2′ passes C2
- **For each *pair of events* figure out the invariant interval** $\Delta s^2 = c^2\Delta t^2 - \Delta x^2$
- **Now apply the invariance** $\Delta s' = \Delta s$

Result: Find each observer thinks the other's clocks are slow ***and improperly synchronized!***

Example: Problem 37-65

Can a long limo fit in a short garage temporarily?



37-65 (cont'd)

(a) Length of car according to Garageman

$$L_c' = \frac{L_c}{\gamma} = L_c\sqrt{1-\beta^2} = (30.5\,\mathrm{m})\,\sqrt{1-(0.9980)^2} = 1.93\ \mathrm{m}\ .$$

(b) Coordinates of event 2 according to Garageman

$$\Delta t_g = t_{g2} - t_{g1} = \frac{L_g - L_c'}{v} = \frac{6.00\,\mathrm{m} - 1.93\,\mathrm{m}}{0.9980(2.998\times10^8\,\mathrm{m/s})} = 1.36\times10^{-8}\ \mathrm{s}\ .$$

(c) Car spends this time inside according to G.

(d) Length of garage according to Carman

$$L_g' = \frac{L_g}{\gamma} = L_g\sqrt{1-\beta^2} = (6.00\,\mathrm{m})\,\sqrt{1-(0.9980)^2} = 0.379\ \mathrm{m}\ .$$

37-65 (cont'd)

(e) Time between events according to Carman

$$\Delta t_c = t_{c1} - t_{c2} = \frac{L_c - L_g'}{v} = \frac{30.5\,\mathrm{m} - 0.379\,\mathrm{m}}{0.9980(2.998\times10^8\,\mathrm{m/s})} = 1.01\times10^{-7}\ \mathrm{s}\ .$$

$$\textit{But}\ t_{C1} = 0 \quad \textit{so}\ t_{C2} = -1.01\times10^{-7}\ s$$

So to Carman, event 2 occurs first, and the car is never entirely inside the garage.