Oscillating Currents

- Ch.30: Induced E Fields: Faraday’s Law
- Ch.30: RL Circuits
- Ch.31: Oscillations and AC Circuits

Review: Inductance

- If the current through a coil of wire changes, there is an induced emf proportional to the rate of change of the current.
- Define the proportionality constant to be the *inductance* $L$: \[ E = -L \frac{di}{dt} \]
- SI unit of inductance is the henry (H).

LC Circuit Oscillations

Suppose we try to discharge a capacitor, using an inductor instead of a resistor:

At time $t=0$ the capacitor has maximum charge and the current is zero.

Later, current is increasing and capacitor’s charge is decreasing.

Oscillations (cont’d)

What happens when $q=0$?

Does $I=0$ also?

No, because inductor does not allow sudden changes.

In fact, $q=0$ means $i=\text{maximum}$!

So now, charge starts to build up on $C$ again, but in the opposite direction!

Textbook Figure 31-1

Energy is moving back and forth between $C,L$.

\[ U_L = U_B = \frac{1}{2} Li^2 \quad U_C = U_E = \frac{1}{2} q^2 / C \]
**Mechanical Analogy**

- Looks like SHM (Ch. 15)  Mass on spring.
- Variable $q$ is like $x$, distortion of spring.
- Then $i = dq/dt$, like $v = dx/dt$, velocity of mass.

By analogy with SHM, we can guess that

$$q = Q \cos(\omega t)$$

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t)$$

**Mathematical description of oscillations**

Note essential terminology: 
*amplitude, phase, frequency, period, angular frequency.* You **MUST** know what these words mean! If necessary review Chapters 10, 15.

**Look at Guessed Solution**

$$q = Q \cos(\omega t)$$

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t)$$

**Get Equation by Loop Rule**

If we go with the current as shown, the loop rule gives:

$$-\frac{q}{C} - L \frac{di}{dt} = 0$$

Now replace $i$ by $dq/dt$ to get:

$$\frac{d^2q}{dt^2} = -\left(\frac{1}{LC}\right)q$$

**Guess Satisfies Equation!**

Start with

$$q = Q \cos(\omega t)$$

so that

$$\frac{dq}{dt} = -\omega Q \sin(\omega t)$$

And taking one more derivative gives us

$$\frac{d^2q}{dt^2} = -\omega^2 Q \cos(\omega t) = -\omega^2 q$$

So the solution is correct, provided our angular frequency satisfies

$$\omega^2 = \left(\frac{1}{LC}\right)$$

**LC Circuit Example**

Given an inductor with $L = 8.0$ mH and a capacitor with $C = 2.0$ nF, having initial charge $q(0) = 50$ nC and initial current $i(0) = 0$.

(a) What is the frequency of oscillations (in Hz)?
(b) What is the maximum current in the inductor?
(c) What is the capacitor’s charge at $t = 30$ μs?
Example: Part (a)

\[ L = 8 \text{ mH}, \ C = 2 \text{ nF}, \ q(0) = 50 \text{ nC}, \ i(0) = 0 \]

(a) What is the frequency of the oscillations?

\[
\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8 \times 10^{-3} \times 2 \times 10^{-9}}} = 2.5 \times 10^5 \text{ rad/s} \\
f = \frac{\omega}{2\pi} = \frac{2.5 \times 10^5 \text{ rad/s}}{6.28 \text{ rad/cycle}} = 4 \times 10^4 = 40 \text{ kHz}
\]

Example: Part (b)

(L = 8 mH, C = 2 nF, q(0) = 50 nC, i(0) = 0)

(b) What is the maximum current?

\[
E = \frac{Q^2}{2C} = \frac{(50 \times 10^{-9})^2}{2 \times 2.0 \times 10^{-9}} = 6.25 \times 10^{-7} \text{ J} \\
\text{But } E = \frac{1}{2} LI^2 \text{ so } I^2 = \frac{2E}{L} \\
\text{So } I = \sqrt{\frac{2E}{L}} = \sqrt{\frac{2 \times 6.25 \times 10^{-7}}{8 \times 10^{-3}}} = 12.5 \text{ mA}
\]

Example: Part (c)

(L = 8 mH, C = 2 nF, q(0) = 50 nC, i(0) = 0)

(c) What is the charge at \( t = 30 \mu s \)?

\[
q(t) = Q \cos(\omega t) \\
= (50 \text{nC}) \cos(2.5 \times 10^5 \times 30 \times 10^{-6}) \\
= (50 \text{nC}) \cos(7.5 \text{ rad}) \\
= (50 \text{nC}) \cos(430^\circ) \\
= 50 \times 0.347 = 17 \text{nC}
\]

AC Voltage Sources

- For an AC circuit, we need an alternating \textit{emf}, or AC power supply.
- This is characterized by its \textit{amplitude} and its \textit{frequency}.

\[ \mathcal{E} = \mathcal{E}_m \sin \omega t \]

\text{amplitude} \hspace{1cm} \text{angular frequency}

AC Currents and Voltages

\[ \mathcal{E} = \mathcal{E}_m \sin \omega t = v_R = V_R \sin \omega t \]

Ohm’s Law gives: \( i_R = v_R / R = (\mathcal{E}_m / R) \sin \omega t \)

So the AC current is: \( i_R = I_R \sin \omega t \)

So the amplitudes are related by: \( V_R = I_R R \)

Notation for oscillating functions

Note that the textbook uses \textit{lower-case} letters for oscillating \textit{time-dependent} voltages and currents, with \textit{upper-case} letters for the corresponding \textit{amplitudes}.

\[ v = V \sin \omega t \hspace{1cm} i = I \sin \omega t \]
AC Voltage-Current Relations

- First apply an alternating emf to a resistor, a capacitor, and an inductor separately, before dealing with them all at once.
- For C, L, define reactance $X$ analogous to resistance $R$ for resistor. Measured in ohms.

\[
V_R = I_R R \\
V_C = I_C X_C \\
V_L = I_L X_L
\]

Summary for R, C, L Separately

<table>
<thead>
<tr>
<th>R</th>
<th>C</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_R = I_R R$</td>
<td>$V_C = I_C X_C$ with $X_C = \frac{1}{\omega C}$</td>
<td>$V_L = I_L X_L$ with $X_L = \omega L$</td>
</tr>
</tbody>
</table>

Q.31-1 Which of the following is true about the phase relation between the current and the voltage for an inductor?

(1) The current is in phase with the voltage.
(2) The current is ahead of the voltage by 90º.
(3) The current is behind the voltage by 90º.
(4) They are out of phase by 180º.

Q.31-2 An inductor $L$ carries a current with amplitude $I$ at angular frequency $\omega$.

What is the amplitude $V$ of the voltage across this inductor?

(1) $V_L = 0.5 V$  
(2) $V_L = 2.0 V$  
(3) $V_L = 6.0 V$  
(4) $V_L = 12 V$  
(5) $V_L = 18 V$  
(6) $V_L = 600 V$
Q.31-2
The reactance is:
\[ X_L = \omega L = 200 \times 0.03 = 6.0 \ \Omega \]
Thus the voltage amplitude is:
\[ V_L = I_L X_L = 3.0 \ \text{A} \times 6.0 \ \Omega = 18 \ \text{V} \]

\begin{align*}
(1) \ V_L &= 0.5 \ \text{V} \\
(2) \ V_L &= 2.0 \ \text{V} \\
(3) \ V_L &= 6.0 \ \text{V} \\
(4) \ V_L &= 12 \ \text{V} \\
(5) \ V_L &= 18 \ \text{V} \\
(6) \ V_L &= 600 \ \text{V} \\
\end{align*}

Impedance and Phase Angle

General problem: if we are given
\[ \mathcal{E}(t) = \mathcal{E}_m \sin \omega t \]
can we find \( i(t) \)?
We can always write
\[ i(t) = I \sin(\omega t - \phi) \]
By definition of impedance
\[ I = \frac{\mathcal{E}_m}{Z} \]

Given \( \mathcal{E}_m \) and \( \omega \), find \( Z \) and \( \phi \).

Series RLC Circuit

1. The currents are all equal.
2. The voltage drops add up to the applied emf as a function of time:
\[ \mathcal{E} = V_R + V_C + V_L \]

BUT: Because of the phase differences, the amplitudes do not add:
\[ \mathcal{E}_m \neq V_R + V_C + V_L \]
So the impedance is not just a sum:
\[ Z \neq R + X_C + X_L \]

Results for series circuits

\[ \mathcal{E}(t) = \mathcal{E}_m \sin \omega t \]
\[ i(t) = I \sin(\omega t - \phi) \]
It turns out that for this particular circuit
\[ Z = \sqrt{R^2 + \left(\frac{X_L - X_C}{R}\right)^2} \]
\[ \tan \phi = \frac{X_L - X_C}{R} \]

Series Circuit Example

Given \( L = 50 \ \text{mH}, \ \ C = 60 \ \mu \text{F}, \ \mathcal{E}_m = 120 \ \text{V}, \ f = 60 \ \text{Hz}, \ \text{and} \ I = 4.0 \ \text{A}. \)

(a) What is the impedance?
(b) What is the resistance \( R \)?

Solution to (a) is easy:
\[ Z = \frac{\mathcal{E}_m}{I} = \frac{120}{4} = 30 \ \Omega \]
Example (part b)  

(a) \( Z = 30 \Omega \)  

L=50 mH, C = 60 \( \mu \)F, \( \varepsilon_m=120 \text{V} \), f=60Hz, I=4.0A  

(b) What is the resistance \( R \)?  

\[ \omega = 2\pi f = 2 \times 3.14 \times 60 = 377 \text{rad/s} \]  
\[ X_L = \omega L = 377 \times 50 \times 10^{-3} = 18.8 \Omega \]  
\[ X_C = \frac{1}{\omega C} = \frac{1}{377 \times 60 \times 10^{-6}} = 44.2 \Omega \]  
\[ X_C - X_L = 44.2 - 18.8 = 25.4 \Omega \]  
\[ R = \sqrt{Z^2 - (X_L - X_C)^2} = \sqrt{30^2 - 25.4^2} = 16 \Omega \]