Oscillating Currents

- **Ch.30: Induced E Fields: Faraday's Law**
- **Ch.30: RL Circuits**
- **Ch.31: Oscillations and AC Circuits**

Review: Inductance

- **If the current through a coil of wire changes, there is an induced emf proportional to the rate of change of the current.**

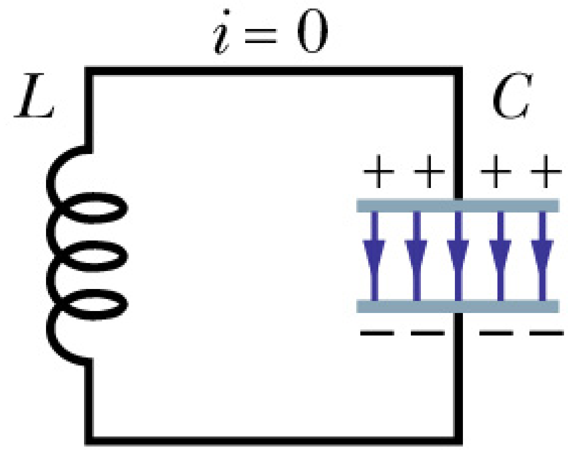
- **Define the proportionality constant to be the *inductance L*:**

$$\boxed{\mathcal{E} = -L\frac{di}{dt}}$$

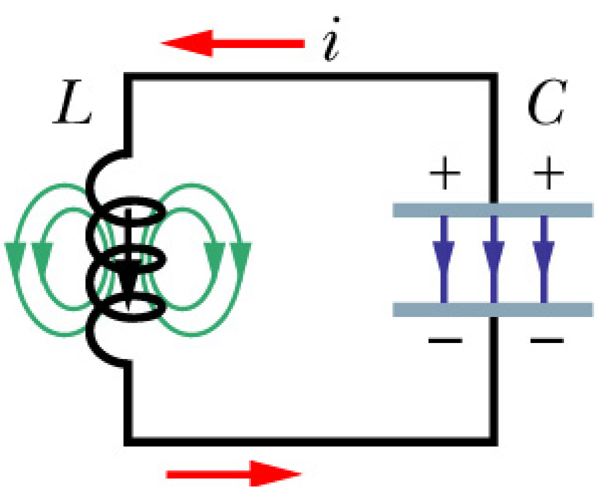
- SI unit of inductance is the **henry** (**H**).

LC Circuit Oscillations

Suppose we try to discharge a capacitor, using an ***inductor*** instead of a resistor:



At time t=0 the capacitor has maximum charge and the current is zero.



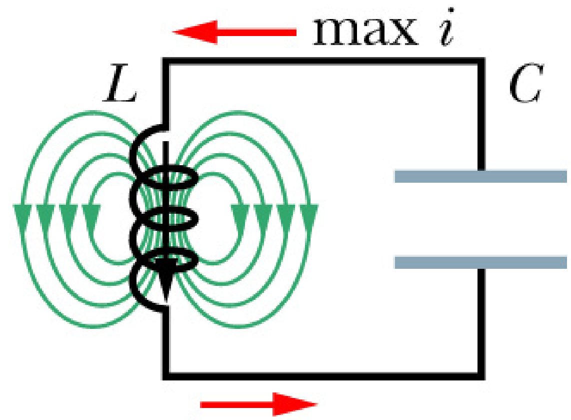
Later, current is increasing and capacitor's charge is decreasing

Oscillations (cont'd)

What happens when q=0?

Does I=0 also?

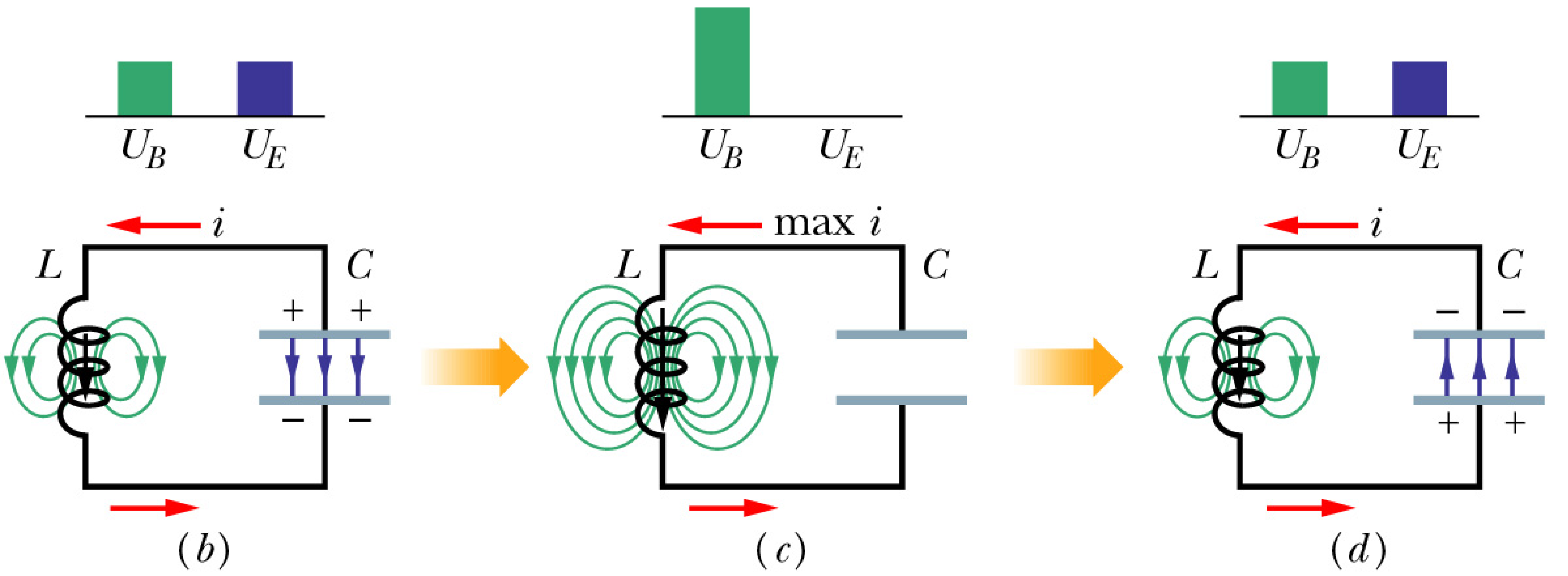
No, because inductor does not allow sudden changes.



In fact, q = 0 means i = maximum!

So now, charge starts to build up on C again, but in the ***opposite direction!***

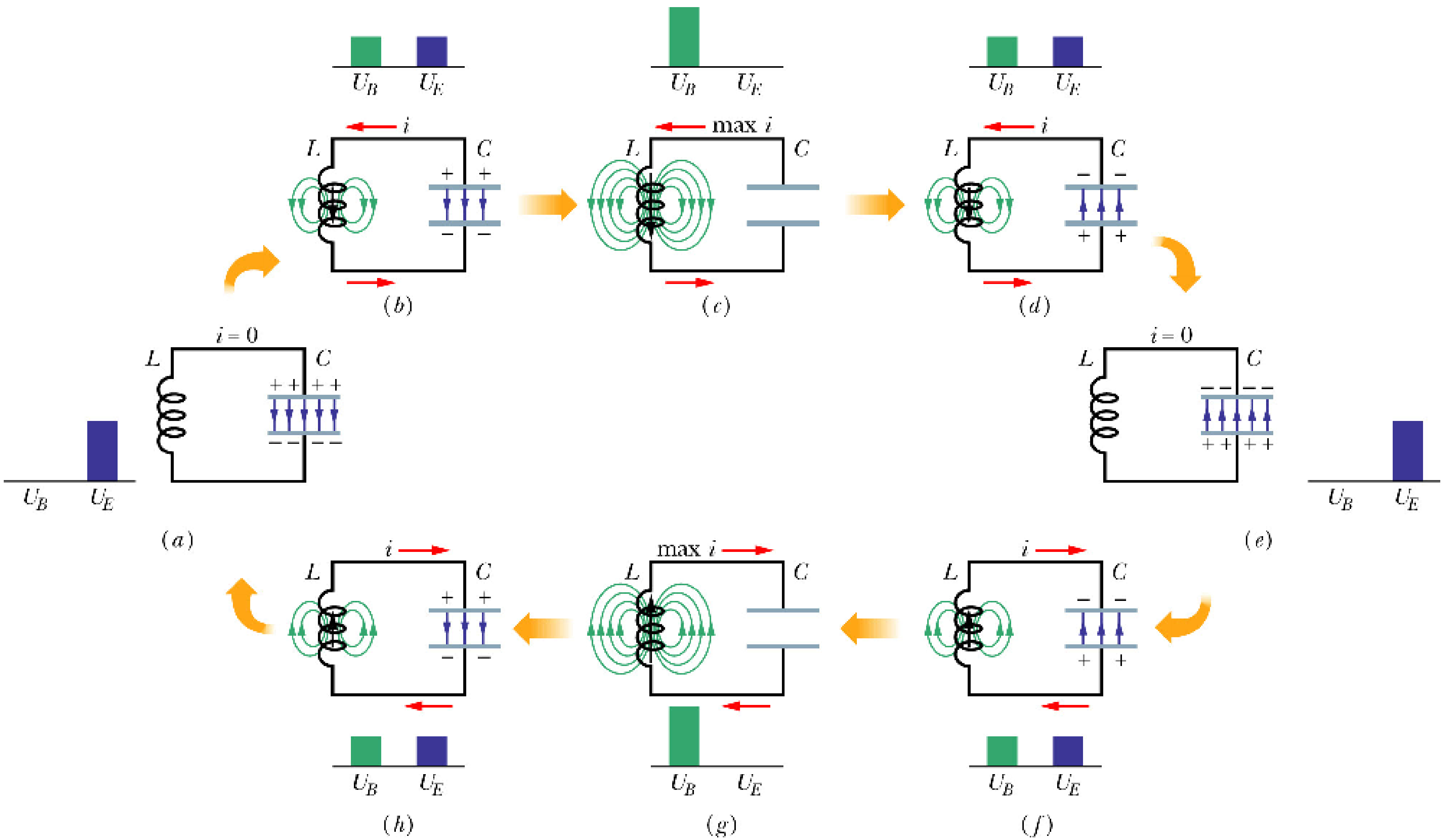
Textbook Figure 31-1



Energy is moving back and forth between C,L

$$U_L = U_B = \frac{1}{2}Li^2 \qquad U_C = U_E = \frac{1}{2}q^2/C$$

Textbook Figure 31-1



Mechanical Analogy

- Looks like SHM (Ch. 15) Mass on spring.
- Variable *q* is like *x*, distortion of spring.
- Then *i=dq/dt*, like *v=dx/dt*, velocity of mass.

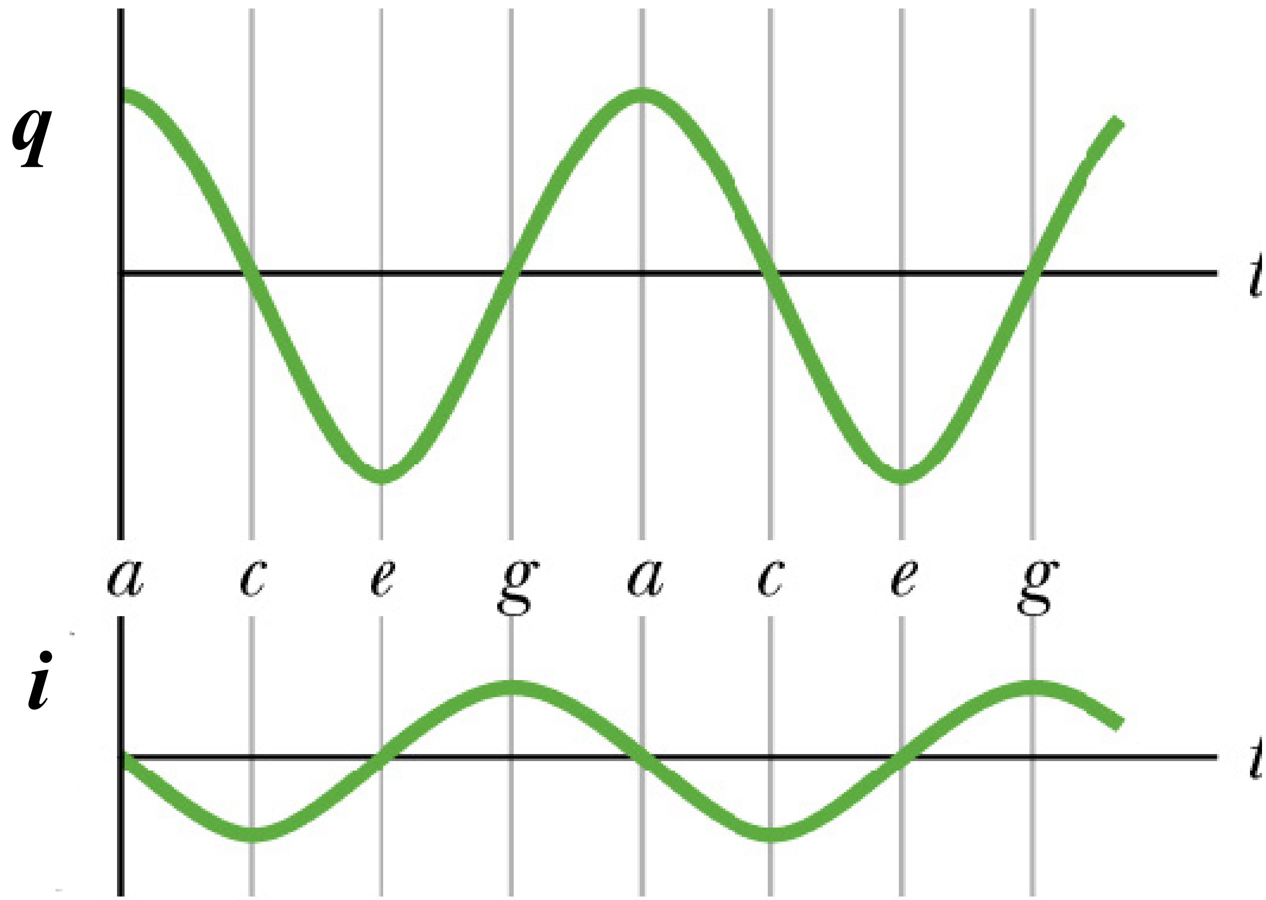
By analogy with SHM, we can guess that

$$q = Q\cos(\omega t)$$

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t)$$

Look at Guessed Solution

$$q = Q\cos(\omega t) \qquad i = \frac{dq}{dt} = -\omega Q\sin(\omega t)$$



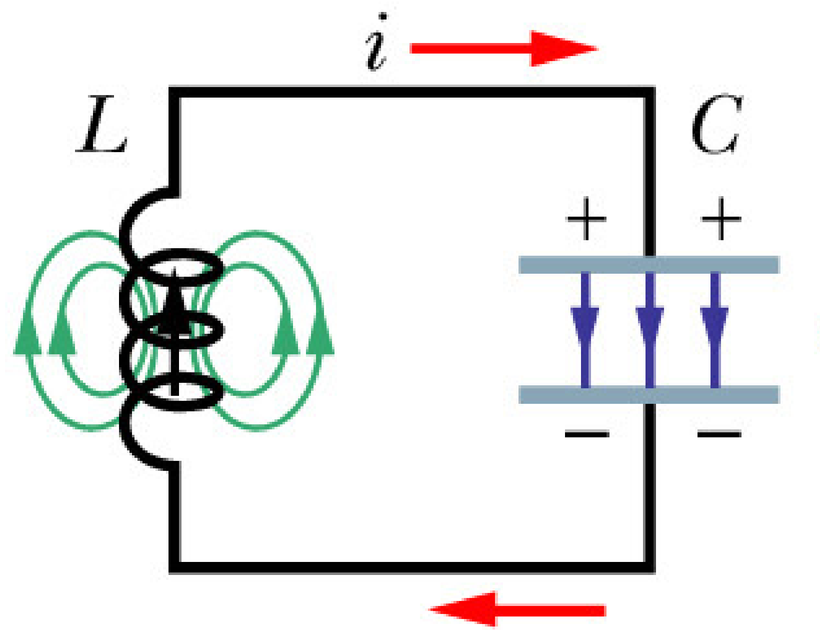
Mathematical description of oscillations

Note essential terminology:
amplitude, phase, frequency, period, angular frequency. You **MUST** know what these words mean! If necessary review Chapters 10, 15.

Get Equation by Loop Rule

If we go with the current as shown, the loop rule gives:

$$-\frac{q}{C} - L\frac{di}{dt} = 0$$



Now replace ***i*** by ***dq/dt*** to get:

$$\boxed{\frac{d^2q}{dt^2} = -\left(\frac{1}{LC}\right)q}$$

Guess Satisfies Equation!

Start with $q = Q\cos(\omega t)$

$$\boxed{\frac{d^2q}{dt^2} = -\left(\frac{1}{LC}\right)q}$$

so that $$\frac{dq}{dt} = -\omega Q\sin(\omega t)$$

And taking one more derivative gives us

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t) = -\omega^2 q$$

So the solution is correct, provided our angular frequency satisfies

$$\boxed{\omega^2 = \left(\frac{1}{LC}\right)}$$

LC Circuit Example

Given an inductor with $L = 8.0$ mH and a capacitor with $C = 2.0$ nF, having initial charge $q(0) = 50$ nC and initial current $i(0) = 0$.

(a) What is the frequency of oscillations (in Hz)?

(b) What is the maximum current in the inductor?

(c) What is the capacitor's charge at $t = 30$ μs?

Example: Part (a)

$$L = 8\ mH,\ \ C = 2\ nF,\ \ q(0) = 50\ nC,\ \ i(0) = 0$$

(a) What is the frequency of the oscillations?

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8\times10^{-3}\times2\times10^{-9}}} = 2.5\times10^{5}\,\frac{rad}{s}$$

$$f = \frac{\omega}{2\pi} = \frac{2.5\times10^{5}\ rad/s}{6.28\ rad/cycle} = 4\times10^{4} = \underline{40\ kHz}$$

Example: Part (b)

(L = 8 mH, C = 2 nF, q(0) = 50 nC, i(0) = 0)

(b) What is the maximum current?

$$E = \frac{Q^2}{2C} = \frac{(50 \times 10^{-9})^2}{2 \times 2.0 \times 10^{-9}} = 6.25 \times 10^{-7}\, J$$

$$But\ E = \frac{1}{2} L I^2 \quad so \quad I^2 = \frac{2E}{L}$$

$$So \quad I = \sqrt{\frac{2E}{L}} = \sqrt{\frac{2 \times 6.25 \times 10^{-7}}{8 \times 10^{-3}}} = \underline{12.5\, mA}$$

Example: Part (c)

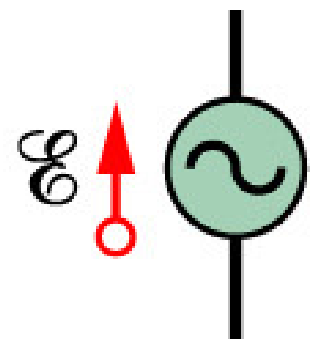
(L = 8 mH, C = 2 nF, q(0) = 50 nC, i(0) = 0)

(c) What is the charge at t = 30 μs?

$$
\begin{aligned}
q(t) &= Q\cos(\omega t) \\
&= (50nC)\cos(2.5\times 10^{5}\times 30\times 10^{-6}) \\
&= (50nC)\cos(7.5\,rad) \\
&= (50nC)\cos(430^{\circ}) \\
&= 50\times 0.347 = \underline{17\,nC}
\end{aligned}
$$

AC Voltage Sources

- **For an AC circuit, we need an** *alternating emf*, **or AC power supply.**
- **This is characterized by its** *amplitude* **and its** *frequency***.**



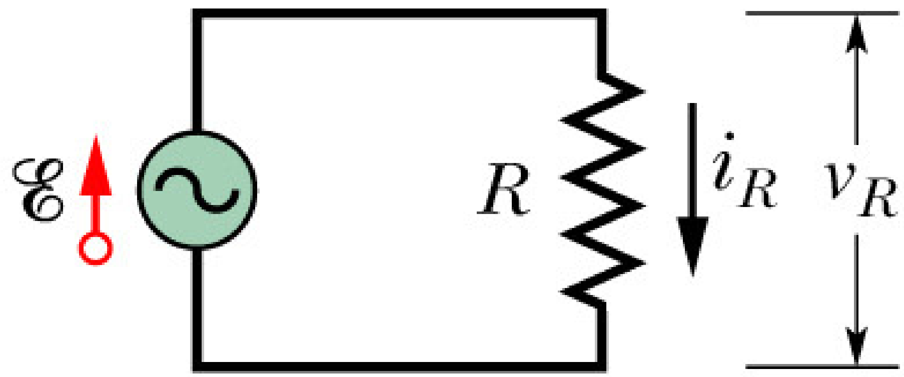
$$\mathcal{E} = \underbrace{\mathcal{E}_m}_{\text{amplitude}} \sin \underbrace{\omega}_{\text{angular frequency}} t$$

Notation for oscillating functions

Note that the textbook uses ***lower-case*** letters for oscillating **time-dependent** voltages and currents, with ***upper-case*** letters for the corresponding **amplitudes**.

$$v = V \sin \omega t \qquad\qquad i = I \sin \omega t$$

AC Currents and Voltages



$$\mathcal{E} = \mathcal{E}_m \sin \omega t = v_R = V_R \sin \omega t$$

Ohm's Law gives: $i_R = v_R / R = (\mathcal{E}_m / R) \sin \omega t$

So the AC current is: $i_R = I_R \sin \omega t$

So the amplitudes are related by: $\underline{V_R = I_R R}$

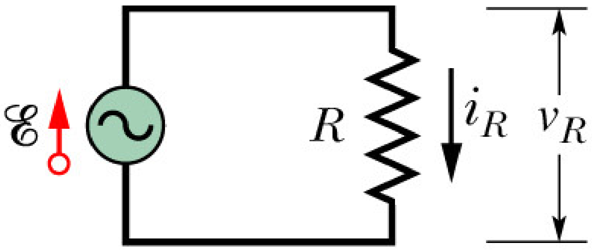
AC Voltage-Current Relations

- **First apply an alternating emf to a resistor, a capacitor, and an inductor** *separately*, **before dealing with them all at once.**
- **For** *C, L,* **define** *reactance X* **analogous to resistance** *R* **for resistor. Measured in ohms.**

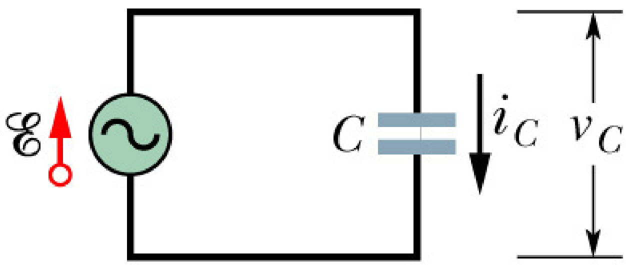
$$V_R = I_R R$$
$$V_C = I_C X_C$$
$$V_L = I_L X_L$$

Summary for R, C, L Separately

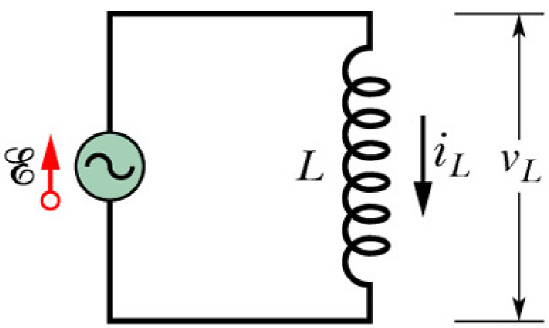
"ELI the ICE man"



$$\begin{cases} V_R = I_R R \\ v_R \text{ and } i_R \text{ are in phase} \end{cases}$$



$$\begin{cases} V_C = I_C X_C \quad \textit{with } X_C = \dfrac{1}{\omega C} \\ i_C \text{ leads } v_C \text{ by } 90°. \end{cases}$$



$$\begin{cases} V_L = I_L X_L \quad \textit{with } X_L = \omega L \\ v_L \text{ leads } i_L \text{ by } 90°. \end{cases}$$

Q.31-1

Which of the following is true about the phase relation between the current and the voltage for an <u>inductor</u>?

(1) The current is in phase with the voltage.

(2) The current is ahead of the voltage by 90^o.

(3) The current is behind the voltage by 90^o.

(4) They are out of phase by 180^o.

Q.31-1

Which of the following is true about the phase relation between the current and the voltage for an inductor?

Remember ELI the ICE man!

Inductor: voltage leads current.

(1) The current is in phase with the voltage.

(2) The current is ahead of the voltage by 90º.

(3) The current is behind the voltage by 90º.

(4) They are out of phase by 180º.

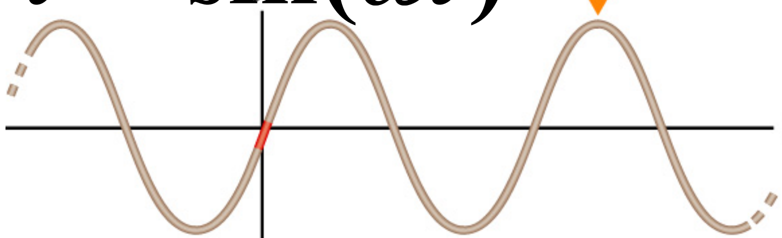
Q.31-1 What is the phase relation between the current and the voltage for an inductor?

Proof:

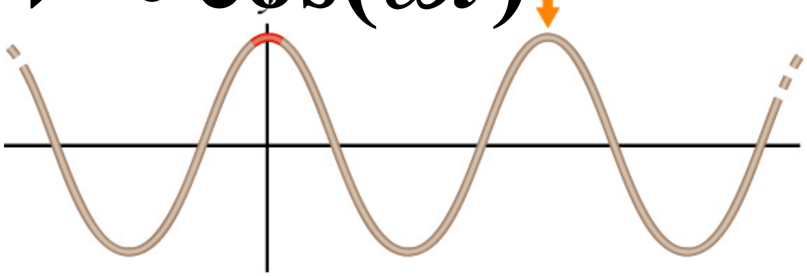
$$Let\ i = I\sin(\omega t)$$

$$Drop\ is\ v = L\frac{di}{dt} = LI\omega\cos(\omega t)$$

$i \propto \sin(\omega t)$

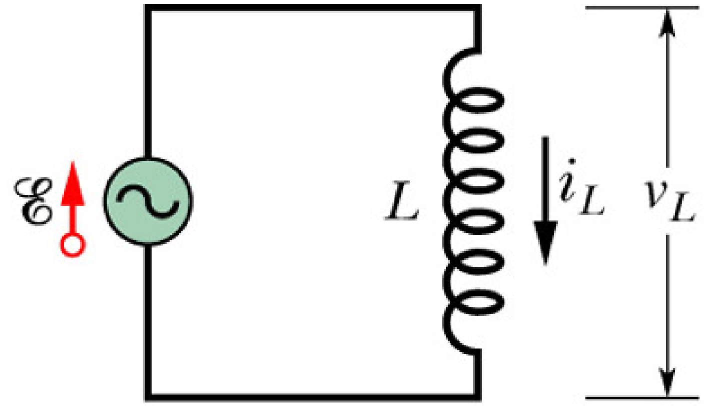


$v \propto \cos(\omega t)$



v hits peak before i

Q.31-2



An inductor L carries a current with *amplitude* I at *angular frequency* ω. What is the *amplitude* V of the voltage across this inductor?

$$I = 3.0\ A$$

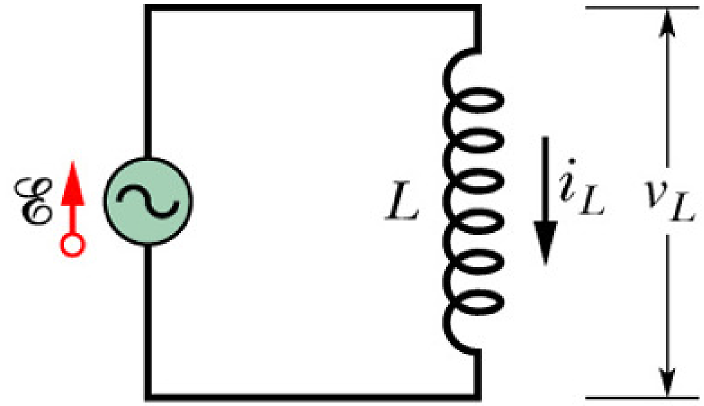
$$\omega = 200\ rad/s$$

$$L = .03\ H$$

(1) $V_L = 0.5\ V$ (2) $V_L = 2.0\ V$ (3) $V_L = 6.0\ V$

(4) $V_L = 12\ V$ (5) $V_L = 18\ V$ (6) $V_L = 600\ V$

Q.31-2



The ***reactance*** is:

$$X_L = \omega L = 200 \times .03 = 6.0\ \Omega$$

$$I = 3.0\ A$$

$$\omega = 200\ rad/s$$

$$L = .03\ H$$

Thus the ***voltage amplitude*** is:

$$V_L = I_L X_L = 3.0\ A \times 6.0\ \Omega = 18\ V$$

(1) $V_L = 0.5\ V$ (2) $V_L = 2.0\ V$ (3) $V_L = 6.0\ V$

(4) $V_L = 12\ V$ (5) $\boxed{V_L = 18\ V}$ (6) $V_L = 600\ V$

Impedance

The "AC Ohm's Law" is: $\boxed{\mathcal{E}_m = IZ}$

where ***Z*** is called the ***impedance.***

Obviously, for a resistor, $Z = R$

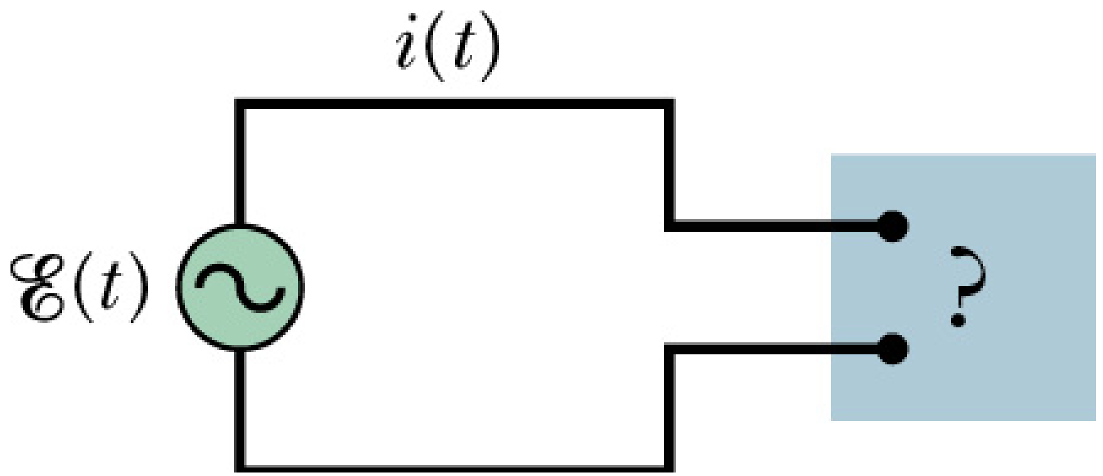
for a capacitor $Z = X_C$ and for an inductor $Z = X_L$

But if you have a combination of circuit elements, *Z* is more complicated.

Impedance and Phase Angle

General problem:
if we are given

$$\boxed{\mathcal{E}(t) = \mathcal{E}_m \sin \omega t}$$



can we find i(t)?

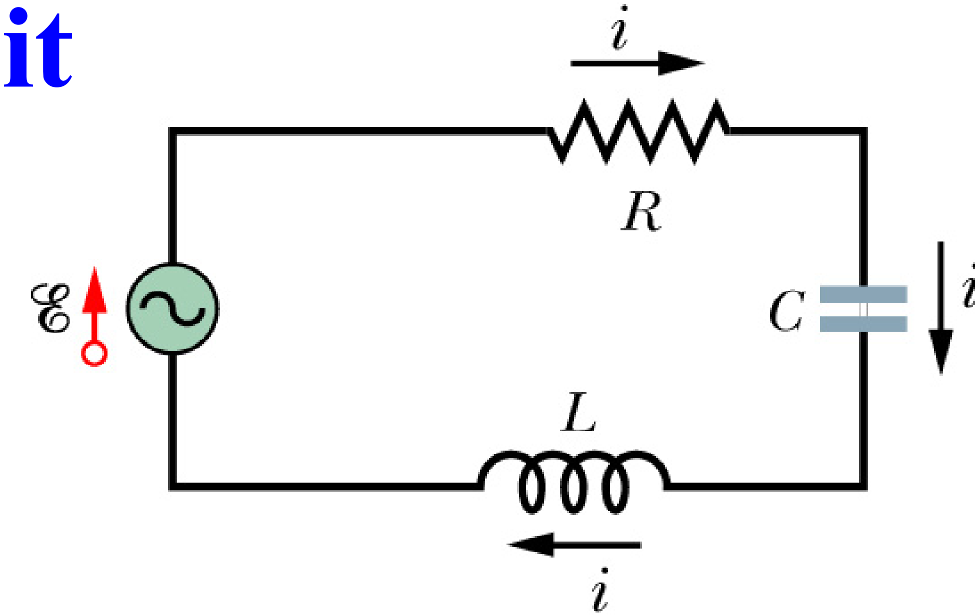
We can always write $\boxed{i(t) = I \sin(\omega t - \phi)}$

By definition of impedance $\boxed{I = \mathcal{E}_m / Z}$

Given ε_m and ω, find Z and ϕ.

??

Series RLC Circuit



1. The currents are all equal.

2. The voltage drops add up to the applied emf as a function of time: $\mathcal{E} = v_R + v_C + v_L$

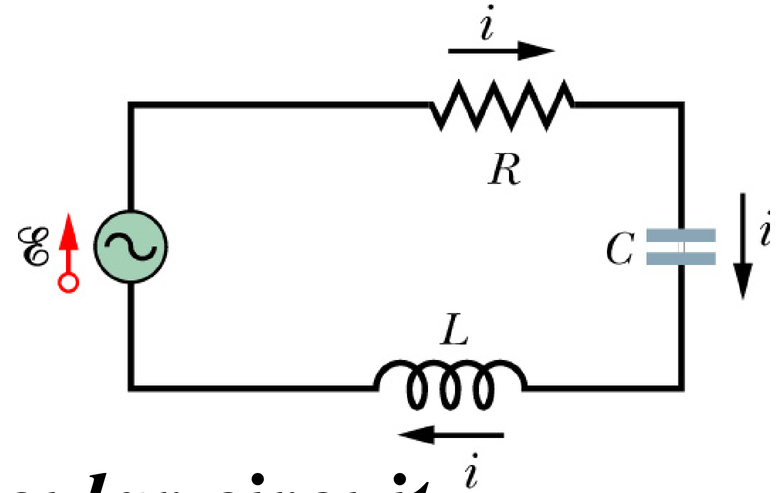
BUT: Because of the ***phase differences***, the **amplitudes** do ***not*** add: $\mathcal{E}_m \neq V_R + V_C + V_L$

So the impedance is not just a sum: $Z \neq R + X_C + X_L$

Results for series circuits

$$\mathcal{E}(t) = \mathcal{E}_m \sin \omega t$$

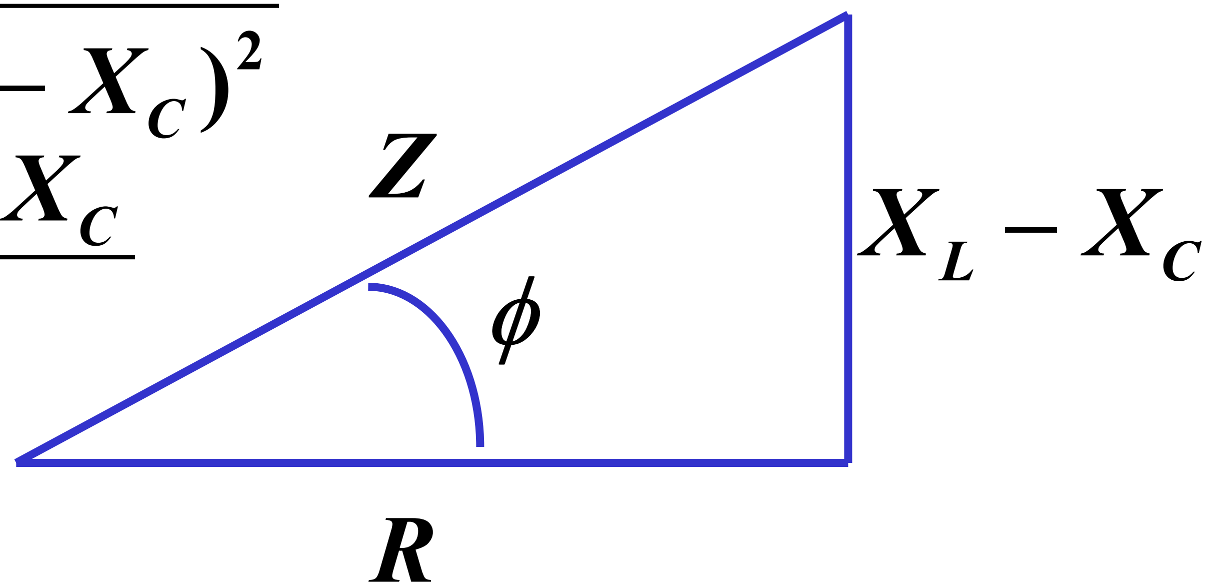
$$i(t) = I \sin(\omega t - \phi)$$



It turns out that for ***this particular circuit***

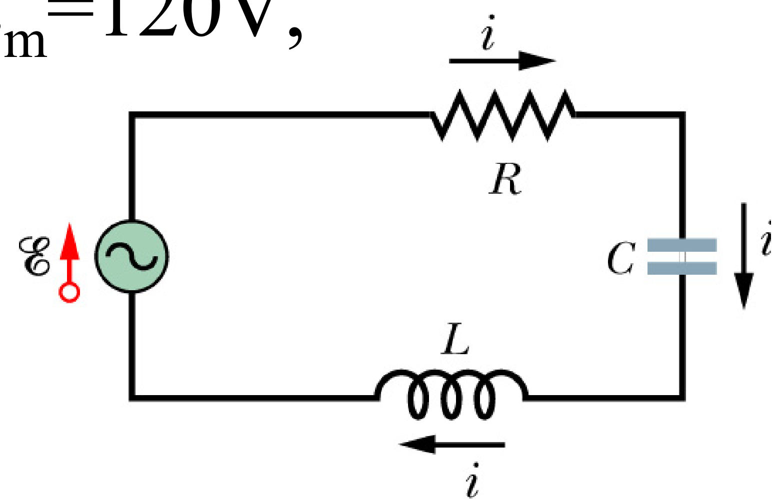
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\tan \phi = \frac{X_L - X_C}{R}$$



Series Circuit Example

Given L = 50 mH, C = 60 μF, ε_m=120V, f=60Hz, and I=4.0A.



(a) What is the impedance?

(b) What is the resistance ***R***?

Solution to (a) is easy:

$$Z = \frac{\mathcal{E}_m}{I} = \frac{120}{4} = \underline{30\,\Omega}$$

Example (part b)

(a) $Z = 30\,\Omega$

L=50 mH, C = 60 μF, ε_m=120V, f=60Hz, I=4.0A

(b) What is the resistance ***R***?

$$\omega = 2\pi f = 2 \times 3.14 \times 60 = 377\, rad/s$$

$$X_L = \omega L = 377 \times 50 \times 10^{-3} = 18.8\,\Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{377 \times 60 \times 10^{-6}} = 44.2\,\Omega$$

$$X_C - X_L = 44.2 - 18.8 = 25.4\,\Omega$$

$$R = \sqrt{Z^2 - (X_L - X_C)^2} = \sqrt{30^2 - 25.4^2} = \underline{16\,\Omega}$$

AC Circuits

- Ch.30: Faraday's Law
- Ch.30: Inductors and RL Circuits:
- Ch.31: AC Circuits

Review: Inductance

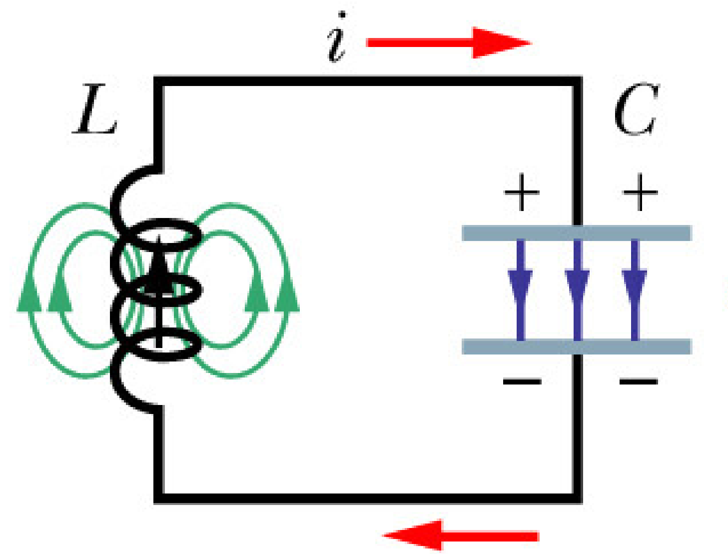
- **If the current through a coil of wire changes, there is an induced emf proportional to the rate of change of the current.**

- **Define the proportionality constant to be the *inductance L*:**

$$\boxed{\mathcal{E} = -L\frac{di}{dt}}$$

- SI unit of inductance is the **henry** (**H**).

Review: LC Circuits



$$q = Q\cos(\omega t)$$

$$i = \frac{dq}{dt} = -\omega Q\sin(\omega t)$$

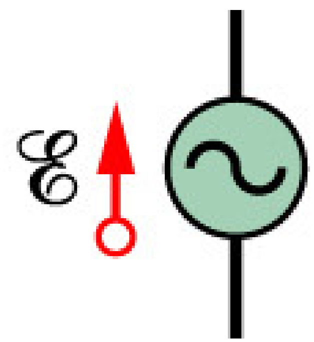
Loop rule gives differential equation: $$\frac{d^2q}{dt^2} = -\left(\frac{1}{LC}\right)q$$

Solution if frequency is correct: $$\omega^2 = \left(\frac{1}{LC}\right)$$

Natural frequency of oscillations!

Review: AC Voltage Sources

- **For an AC circuit, we need an** *alternating emf*, **or AC power supply.**
- **This is characterized by its** *amplitude* **and its** *frequency*.



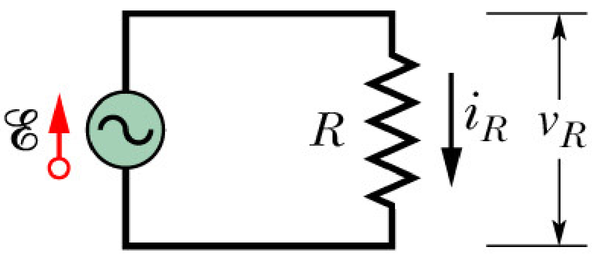
$$\mathcal{E} = \mathcal{E}_m \sin \omega t$$

amplitude

angular frequency

Review: R, C, L Separately

"ELI the ICE man"



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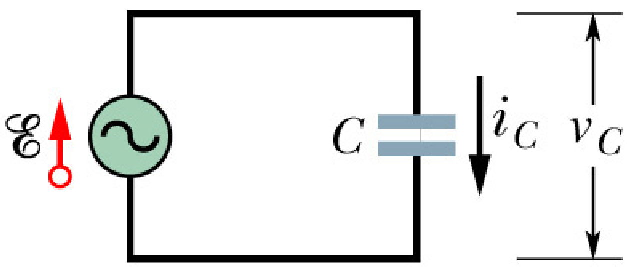

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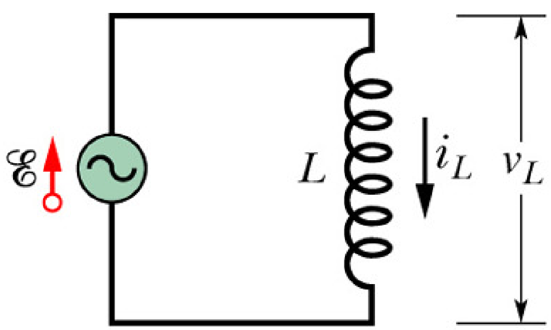
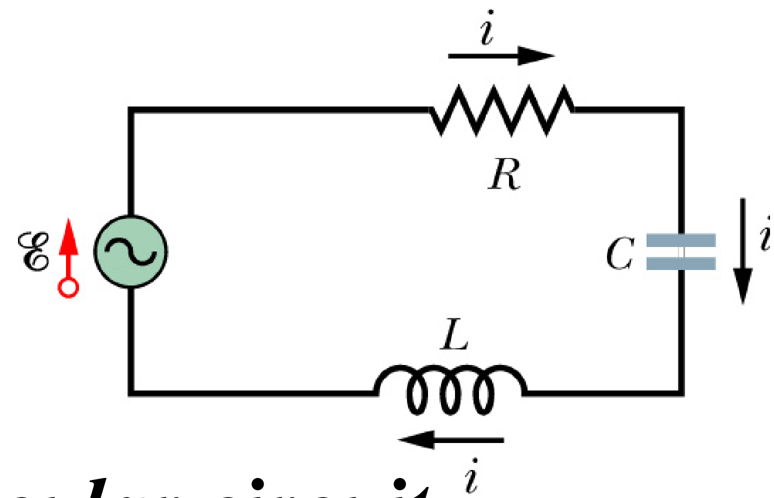


$$\begin{cases} V_L = I_L X_L \quad \text{with } X_L = \omega L \\ v_L \text{ leads } i_L \text{ by } 90°. \end{cases}$$

Review: Simple series circuit

$$\mathcal{E}(t) = \mathcal{E}_m \sin \omega t$$

$$i(t) = I \sin(\omega t - \phi)$$



It turns out that for ***this particular circuit***

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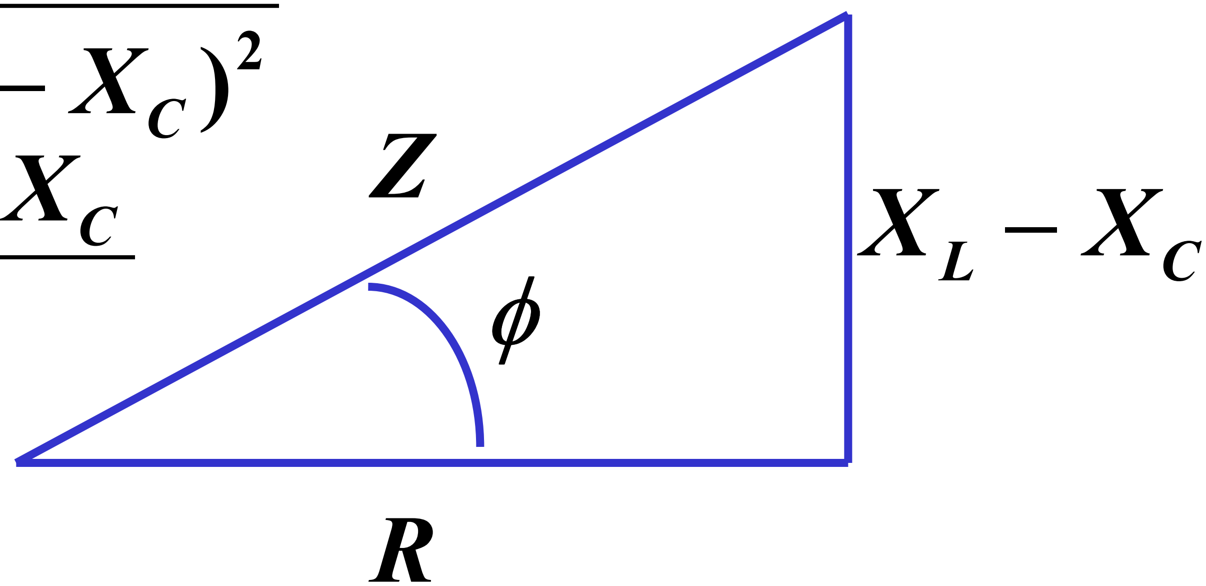

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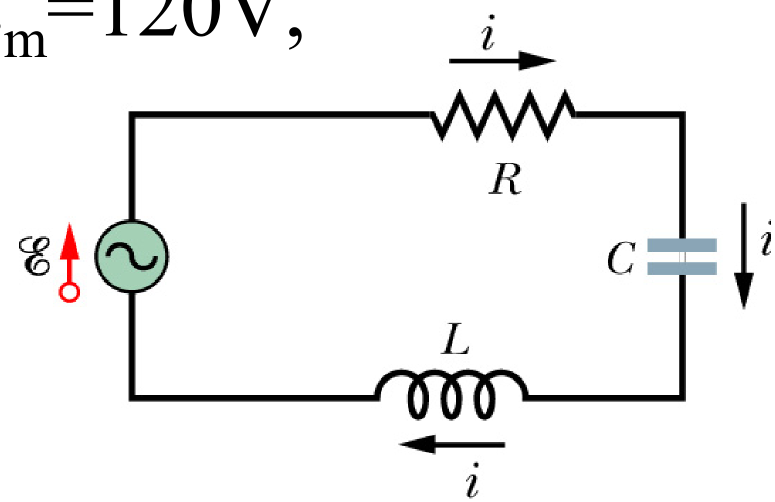

(a) What is the impedance?

(b) What is the resistance ***R***?

Solution to (a) is easy:

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Example (part b)

(a) $Z = 30\,\Omega$

L=50 mH, C = 60 μF, ε_m=120V, f=60Hz, I=4.0A

(b) What is the resistance ***R***?

$$\omega = 2\pi f = 2 \times 3.14 \times 60 = 377\, rad/s$$

$$X_L = \omega L = 377 \times 50 \times 10^{-3} = 18.8\,\Omega$$

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$$X_C - X_L = 44.2 - 18.8 = 25.4\,\Omega$$

$$R = \sqrt{Z^2 - (X_L - X_C)^2} = \sqrt{30^2 - 25.4^2} = \underline{16\,\Omega}$$

Q.31-3

A certain series RLC circuit is driven by an applied emf with angular frequency 20 radians per second. If the maximum charge on the capacitor is 0.03 coulomb, what is the maximum current in the circuit?

(1) 6 A (2) 1.5 A (3) 0.6 A (4) 0.15 A

(5) Not enough information

Q.31-3

A certain series RLC circuit is driven by an applied emf with angular frequency 20 radians per second. If the maximum charge on the capacitor is 0.03 coulomb, what is the maximum current in the circuit?

$$q = Q\sin(\omega t) \qquad i = \frac{dq}{dt} = \omega Q\cos(\omega t)$$

$$I = \omega Q = 20 \times 0.03 = 0.6\ A$$

(1) 6 A (2) 1.5 A $\boxed{\text{(3) 0.6 A}}$ (4) 0.15 A

(5) Not enough information

Q.31-4

In a series RLC circuit, find the resistance, given the impedance and phase constant:

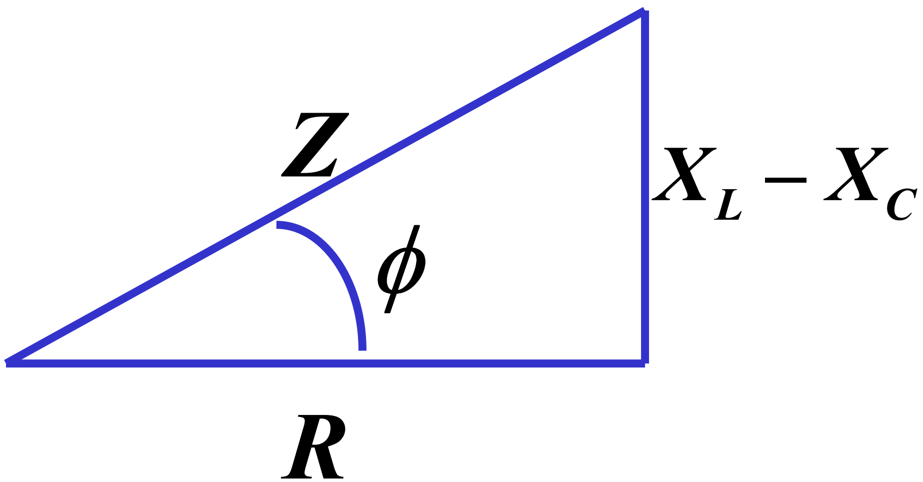
$Z = 500\ \Omega$ $\quad\quad \phi = 60°$ $\quad\quad R = ?$

(1) 400 Ω (2) 350 Ω (3) 300 Ω (4) 250 Ω

Q.31-4

In an RLC circuit:

Z = 500 Ω $\qquad$ **ϕ = 60°** $\qquad$ **R = ?**



$$R = Z\cos\phi$$

$$\cos(60°) = \sin(30°) = \frac{1}{2}$$

$$R = Z/2 = 250\ \Omega$$

(1) 400 Ω $\qquad$ (2) 350 Ω $\qquad$ (3) 300 Ω $\qquad$ $\boxed{\text{(4) 250 Ω}}$

Resonance

For a given series RLC circuit, what applied frequency will give the biggest current?

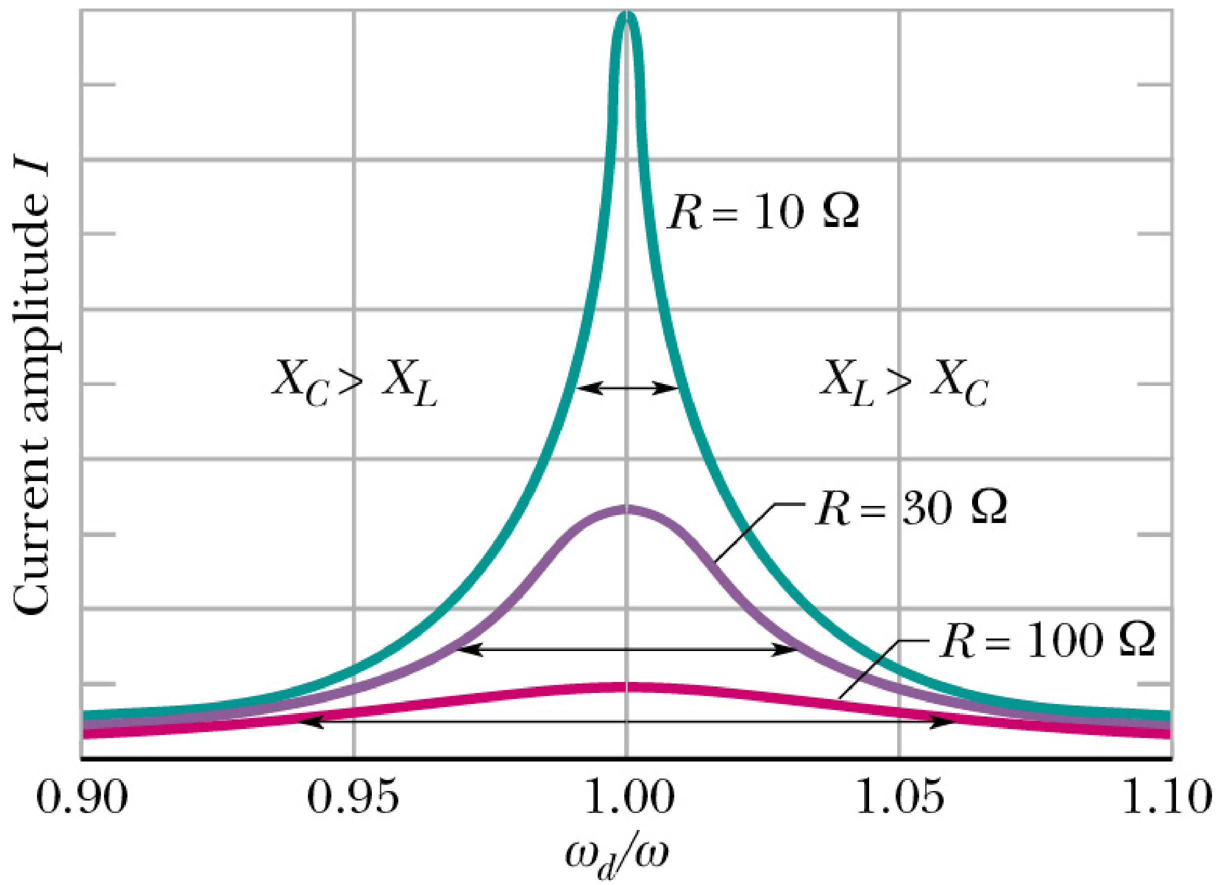
$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

I is biggest when denominator is smallest, which is when reactances cancel:

$$\omega L = \frac{1}{\omega C}$$ Which gives $$\omega^2 = \frac{1}{LC}$$

Same as natural frequency for oscillations!

Resonance Plot: I *vs* ω



Peak at $\omega = \mathbf{1}/\sqrt{LC}$ Becomes sharper as $R \to 0$

Power in AC Circuits

To know how much power will be consumed in an AC circuit we need more than the impedance $\boldsymbol{Z}$; we also need the phase angle ϕ.

AC Power

What is the power provided by an AC source?

- Only interested in the ***time average!***
- Time average power to L and C is ***zero.***
- So P_{ave}(from source) = P_{ave}(to resistor).

$$P_{ave} = \langle v_R i_R \rangle = \langle (V_R \sin \omega t)(I_R \sin \omega t) \rangle$$

$$= V_R I_R \langle \sin^2 \omega t \rangle = \frac{1}{2} V_R I_R$$

RMS Values

$$V_{RMS} = \sqrt{\langle v^2 \rangle} = \sqrt{\langle V^2 \sin^2 \omega t \rangle} = V\sqrt{\frac{1}{2}} = V/\sqrt{2}$$

$$\textit{Likewise} \quad I_{RMS} = \sqrt{\langle i^2 \rangle} = I/\sqrt{2}$$

So for a resistor:

$$P_{ave} = \underline{\underline{V_{RMS} I_{RMS}}} = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}} = \frac{1}{2}VI$$

Power and Phase Angle

Power to resistor $$P = i_R{}^2 R = I_{R^2} R \sin^2(\omega t - \phi)$$

$$P_{avg} = I_R{}^2 R \left\langle \sin^2 \right\rangle_{avg} = \frac{1}{2} I_R{}^2 R$$

But we want result in terms of impedance and phase angle:

$$R = Z \cos\phi$$

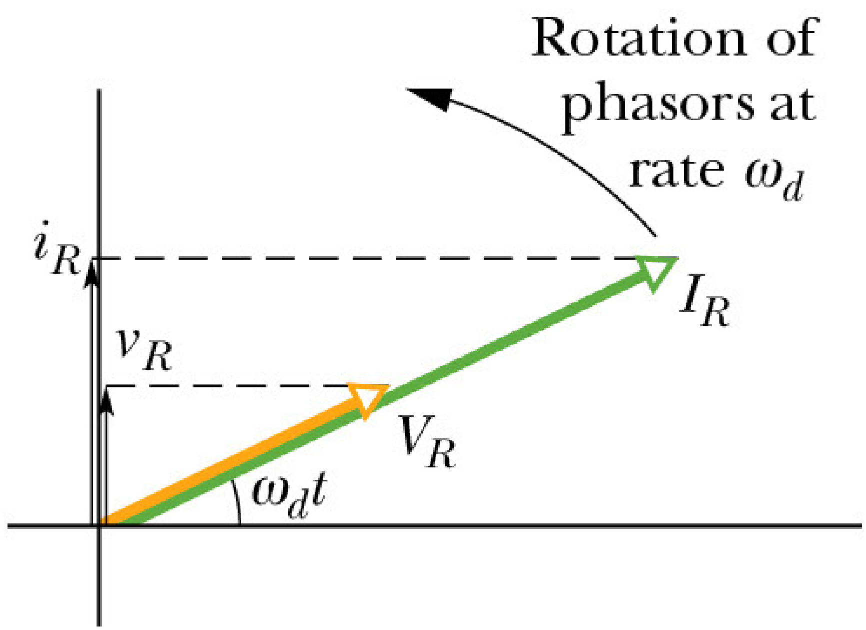
$$So \quad P_{avg} = \frac{1}{2} I^2 Z \underbrace{\cos\phi}_{\textit{Power factor}}$$

What about more complicated circuits?

Method of Phasors

- Useful mathematical trick for working with oscillating functions having different phase relationships.
- We'll use it to derive the known result for the series circuit, to show how it works. Good for general case.

$$\boldsymbol{i_R = I_R \sin(\omega t)}$$



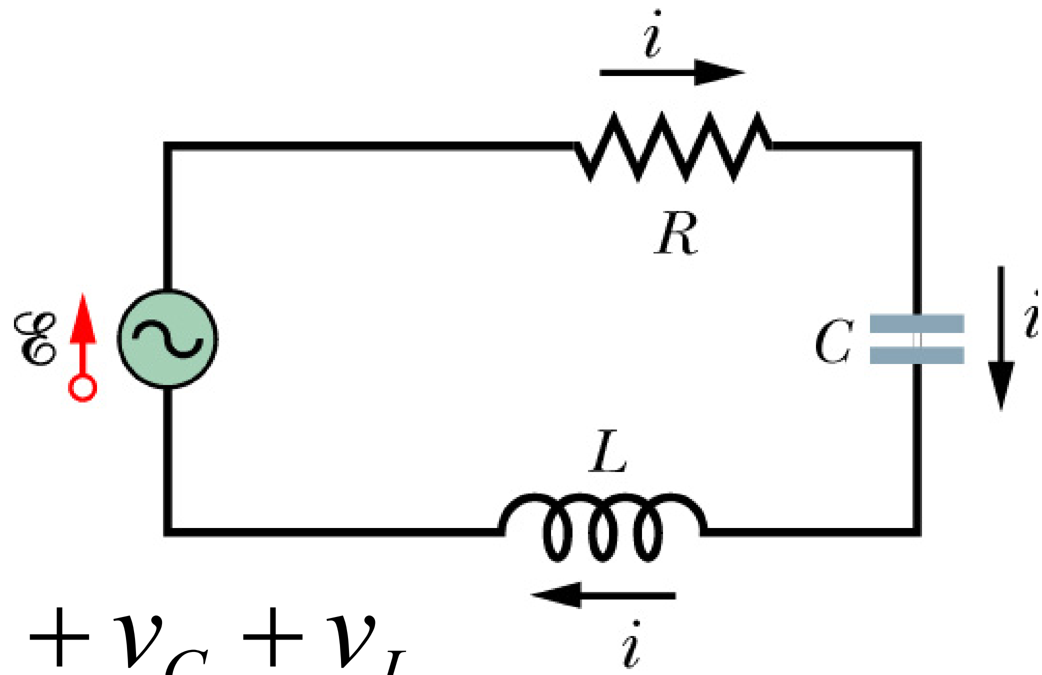
$I_R =$ Length of phasor

$i_R =$ Vertical component of phasor

Series RLC Circuit

1. The currents are all equal.

2. The voltage drops add up to the applied emf:



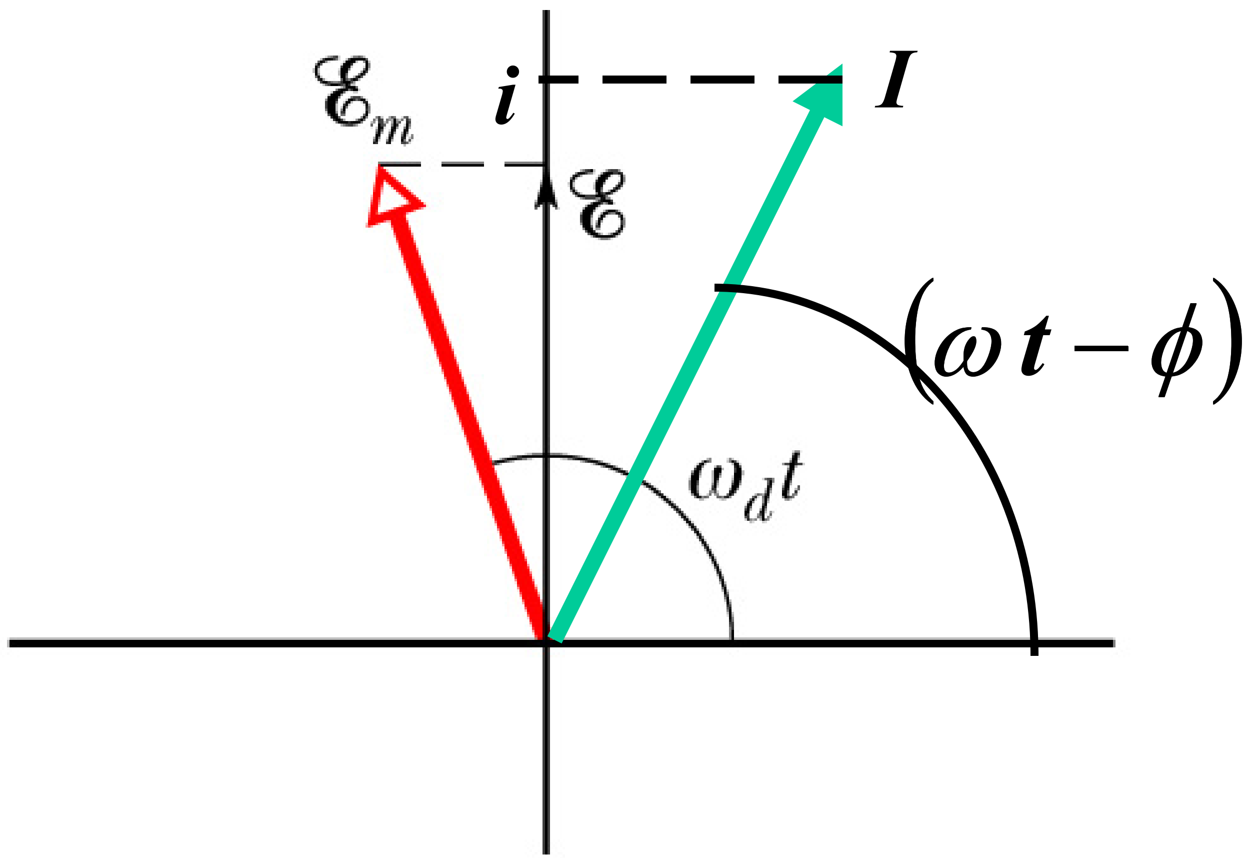
$$\mathcal{E} = v_R + v_C + v_L$$

Because of the phase differences, the amplitudes do not add, **but** in the phasor diagram, this means that the vertical components **do** add, so we get the right answer if we add the phasors **as vectors.**

Phasors for Series RLC Circuit

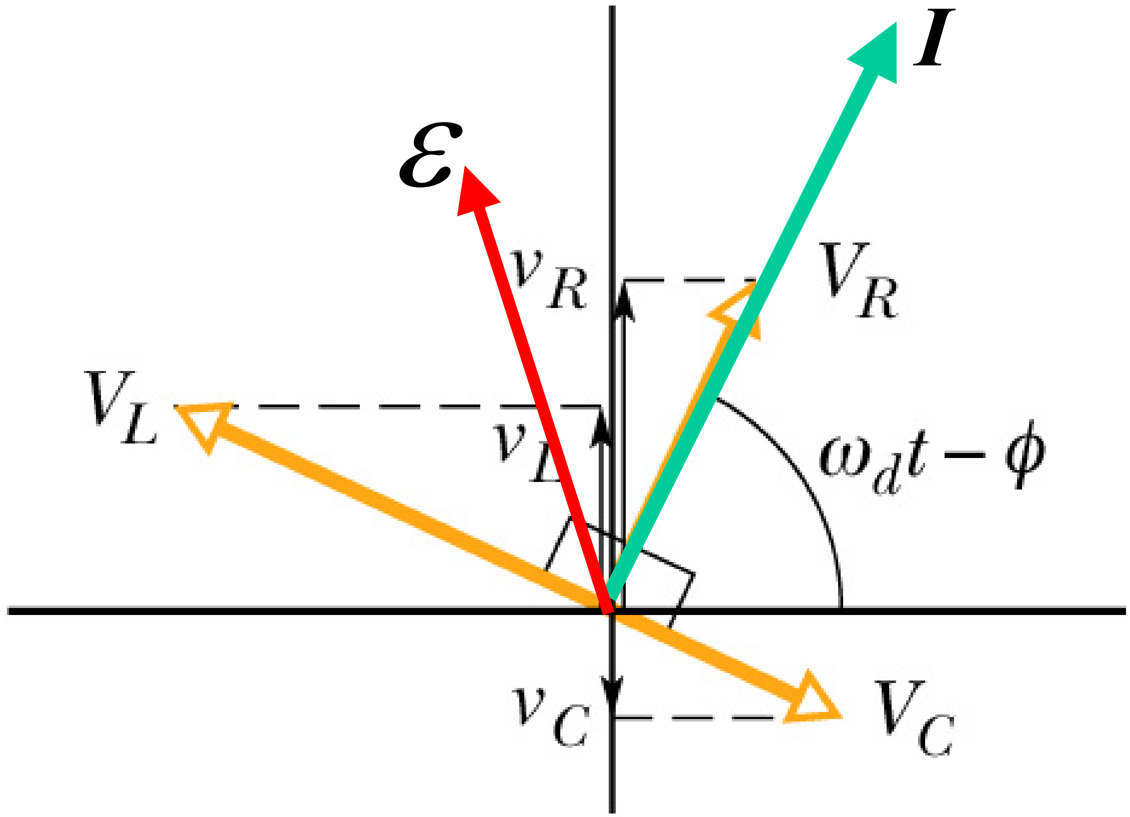
$$\mathcal{E} = v_R + v_C + v_L = \mathcal{E}_m \sin \omega t$$

$$i_R = i_C = i_L = I \sin(\omega t - \phi)$$



Adding the Voltage Phasors

$$\mathcal{E} = v_R + v_C + v_L = \mathcal{E}_m \sin \omega t$$



Adding the Voltage Phasors

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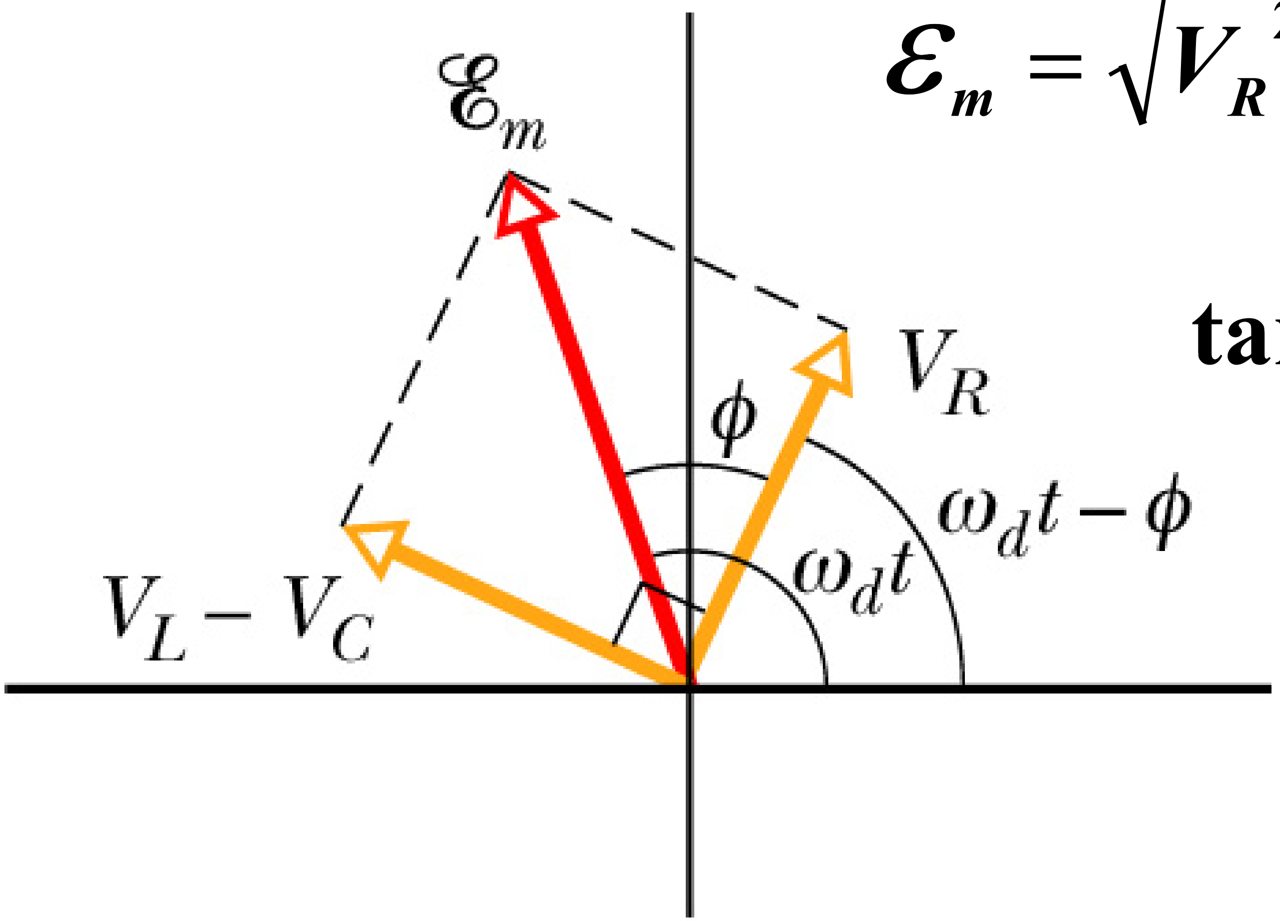

$$\mathcal{E}_m = \sqrt{V_R^{\ 2} + (V_L - V_C)^2}$$

$$\tan \phi = \frac{V_L - V_C}{V_R}$$

Impedance and Phase Angle

$$\mathcal{E}_m = \sqrt{{V_R}^2 + (V_L - V_C)^2} \qquad \tan\phi = \frac{V_L - V_C}{V_R}$$

Divide by the common current amplitude ***I:***

$$\mathcal{E}_m = IZ, \quad V_R = IR, \quad V_L = IX_L, \quad V_C = IX_C$$

$$\boxed{\begin{gathered} Z = \sqrt{R^2 + (X_L - X_C)^2} \\ \tan\phi = \frac{X_L - X_C}{R} \end{gathered}}$$