Potentials and Fields

Review: Definition of Potential

- *Potential* **is defined as** *potential energy* **per unit charge. Since change in potential energy is work done, this means**

$$\Delta V = -\int E_x dx \qquad \text{and} \qquad E_x = -\frac{dV}{dx} \quad etc.$$

$$\boxed{\text{The potential difference between any two points is the work required to carry a unit positive test charge between those two points.}}$$

Review: Coulomb's Law

So we now have a ***third form of Coulomb's Law:***

1. $F = kQq / r^2$
2. $E = kQ / r^2$
3. $V = kQ / r$

Review: Basics about Potentials

- $U = qV$
- $V = kQ/r$
- $W_{AB} = q\,(V_B - V_A)$

$$\Delta V = -\int E_x dx$$

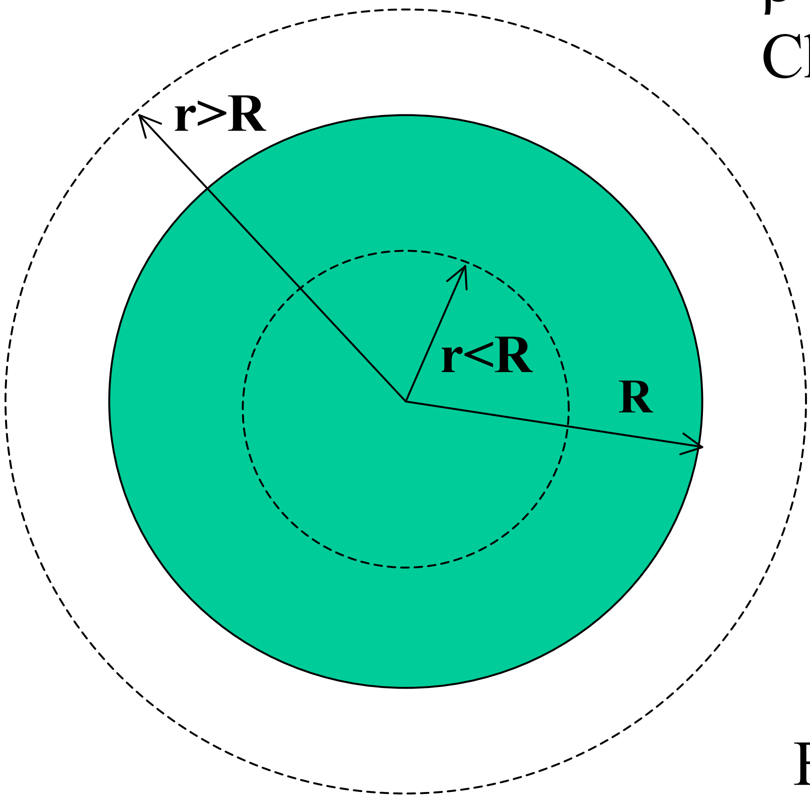
$$E_x = -\frac{dV}{dx} \quad etc.$$

Field of a Uniformly Charged Sphere

Good example for main ideas of this week.

- **Given a non-conducting sphere with a uniform volume charge density.**
- **Use *Gauss's Law* to find the electric field inside the sphere. (Ch. 23)**
- **Side effect: again prove the shell theorem!**
- **Use that solution to *find potential* inside and outside this sphere. (Ch. 24)**

Uniformly charged sphere



ρ= volume charge density=
Charge/Volume= Coulomb/Meter

Total charge

$$Q = \frac{4\pi}{3} R^3 \rho$$

Field outside

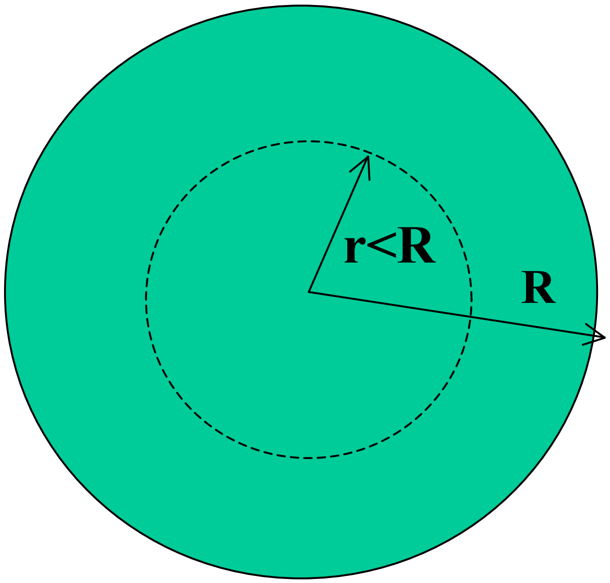
$$E = \frac{Q}{\varepsilon_0 r^2}, \ r > R$$

Field inside

$$Q \Rightarrow Q(r) = \frac{4\pi}{3} r^3 \rho, \quad E = \frac{Q(r)}{\varepsilon_0 r^2} = \frac{4\pi}{3\varepsilon_0} \rho r, \ r < R$$

Q.23-3

If the uniformly charged sphere has radius R=2 cm and total charge Q = 24 nC, how much charge Q_{in} lies within r = 1 cm of the center?



1) **12 nC**
2) **8 nC**
3) **6 nC**
4) **4 nC**
5) **3 nC**

Q.23-3

Solution:

Volume of sphere is $\left(Vol.\right)=\frac{4}{3}\pi r^3$

So charge inside is $Q=(Vol.)\rho=\frac{4}{3}\pi r^3\rho$

So charge within a sphere is proportional to the radius cubed: $\boxed{Q \propto r^3}$

So if sphere with r=1 has (1/2)3 = 1/8 the volume, and 1/8 the charge.

$$Q_{in}=(1/8)Q_{tot}=\frac{24\ nC}{8}=3\ nC$$

Field due to solid charged sphere.

So the field inside is:

$$E = \frac{\rho r}{3\varepsilon_0}$$

And the field outside is:

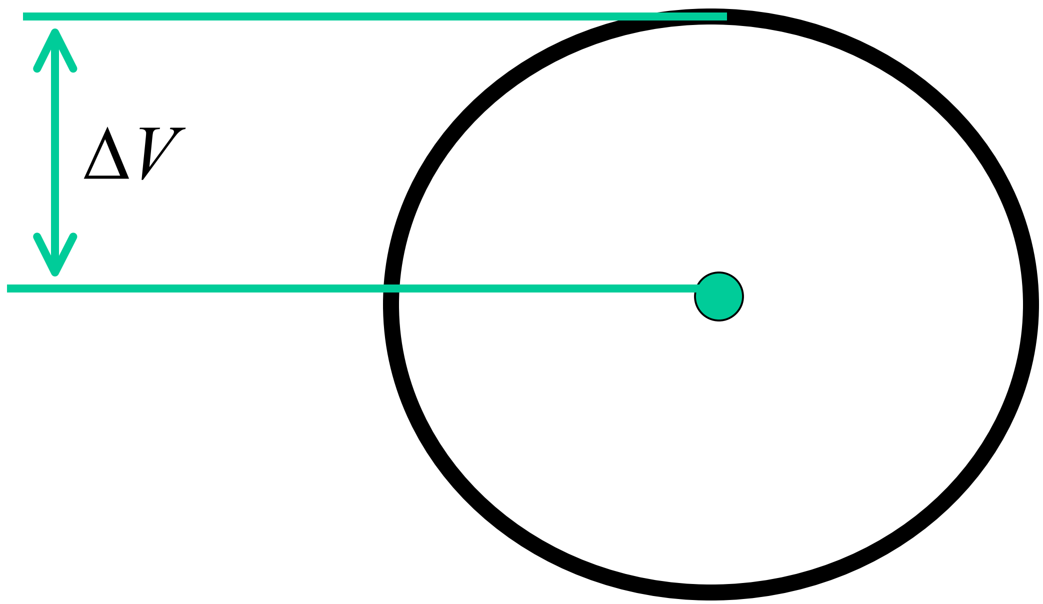
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Exercise for the student:

Prove these two formulas agree when r=R.

Hint: use $Q = \frac{4}{3}\pi R^3 \rho$

Potential Difference



Continuing with the uniformly charged sphere problem.

- What is the potential difference between the center of the sphere and the surface?
- How much work would it require to move an electron *from* the center *to* the surface?

By definition $$\Delta V = -\int_R^0 \vec{E} \cdot d\vec{r} = \int_0^R E \, dr$$

But we already know $$E = \frac{\rho r}{3\varepsilon_0}$$

Therefore

$$\Delta V = \int_0^R \frac{\rho r}{3\varepsilon_0} dr = \frac{\rho}{3\varepsilon_0} \int_0^R r \, dr = \frac{\rho}{3\varepsilon_0} \frac{R^2}{2} = \frac{\rho R^2}{6\varepsilon_0}$$

Suppose we have some given numbers:

$$R = 3\ cm \qquad \rho = 3 \times 10^{-6}\ C/m^3$$

(So that $Q = \frac{4}{3}\pi R^3 \rho = 3.4 \times 10^{-10}\ C = 0.34\ nC$)

Then

$$\Delta V = \frac{\rho R^2}{6\varepsilon_0} = \frac{3 \times 10^{-6} \times (3 \times 10^{-2})^2}{6 \times 9 \times 10^{-12}} = \underline{50\ V}$$

Q.24-3 **Using the same numbers, what is the potential at the surface of this sphere?**

$$R = 3\ cm$$

$$\rho = 3 \times 10^{-6}\ C/m^3$$

$$Q = 3.4 \times 10^{-10}\ C$$

1) 250 V
2) 100 V
3) 50 V
4) 25 V
5) 0

Q.24-3

Outside, E is given by Coulomb's Law (shell theorem), so work to bring in test charge from infinity is same as for point charge:

$$E = kQ/r^2 \qquad V = kQ/r$$

$$V = kQ/R = \frac{\left(9\times10^{9}\right)\left(3.4\times10^{-10}\right)}{3\times10^{-2}} = 102\,V$$

1) 250 V
2) 100 V
3) 50 V
4) 25 V
5) 0

A new unit of energy

- One ***electron-volt*** (eV) is the energy to move an electron between points with a potential difference of one volt.
- This is not an SI unit but is universally used in discussing processes involving electrons, atoms, nano-scale structures, etc.

$$U = qV$$

So if V=1 V and q=e then:

$$U = e \times V = (1\,V)(e) = \underline{1\,eV}$$

$$= (1\,V) \times (1.6 \times 10^{-19}\ C) = \underline{1.6 \times 10^{-16}\ J}$$

A new unit of energy

$$1\,eV = 1.6\times10^{-19}\ J$$

So in our previous example, where we have V=50 V and q=e then the work required to move an electron from the center to the surface is:

$$U = qV = (50\ V)(e) = 50\ eV$$

$$= (1.6\times10^{-19}\ C)\times(50\ V) = 8\times10^{-18}\ J$$

Much simpler to just give 50 eV as the answer.

More about units

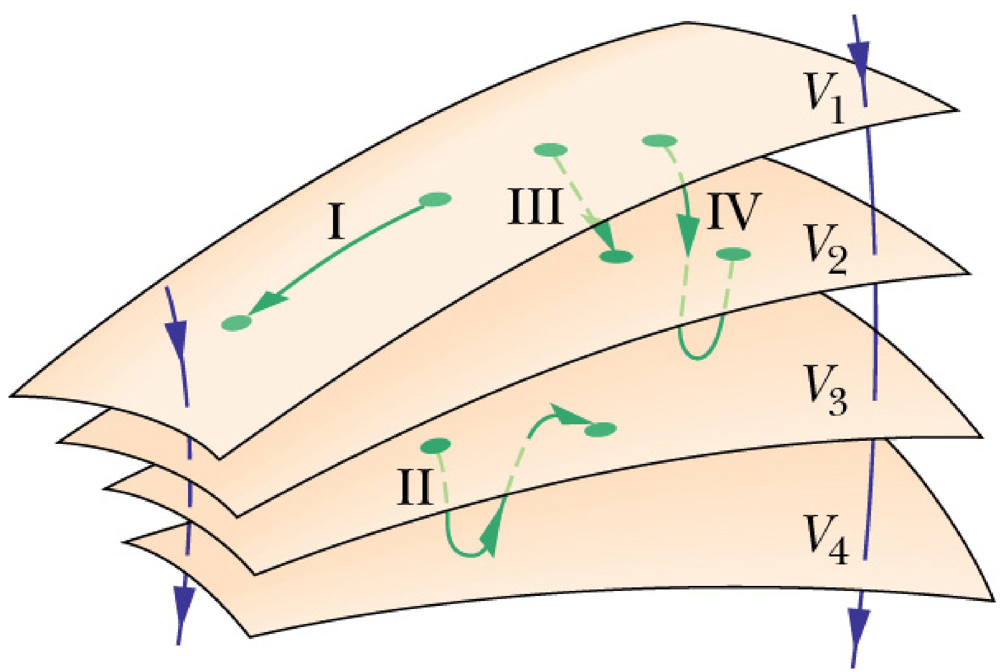
- The SI unit for potential is the volt.
- The SI unit for electric field is N/C.
- But $E = dV/dx$ so another unit for E could be volts/meter.

It is correct to say that the SI unit for E is V/m.

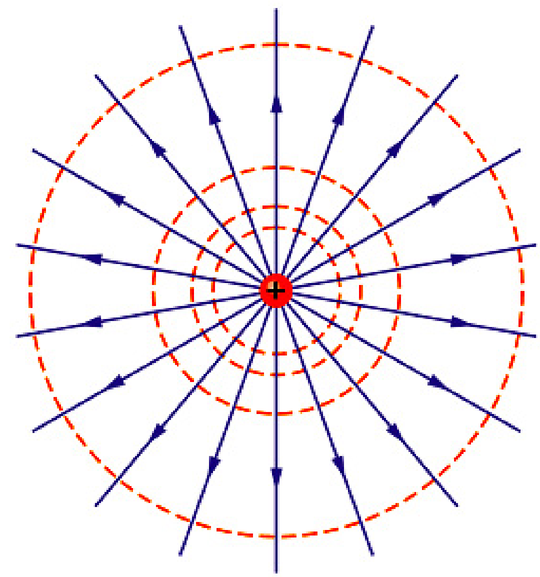
This is more commonly used in applications.

Equipotential surfaces

Any surface which is everywhere ***perpendicular*** to the electric field is an equipotential surface. The potential has the same value everywhere on this surface. This is true because no work is required to move a test charge from place to place on the surface.

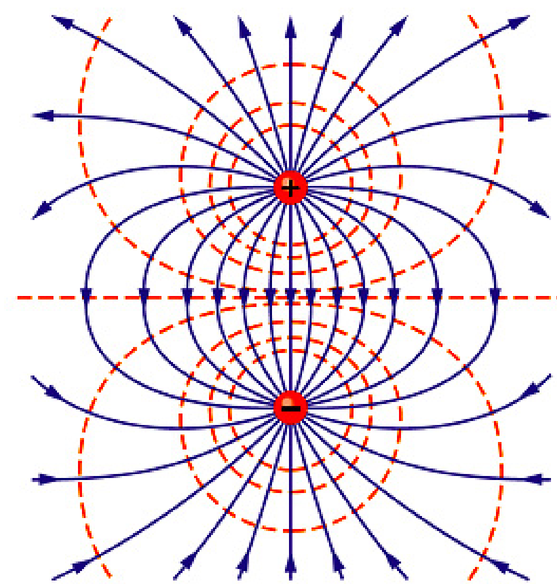


For a point charge, equipotentials are spheres:



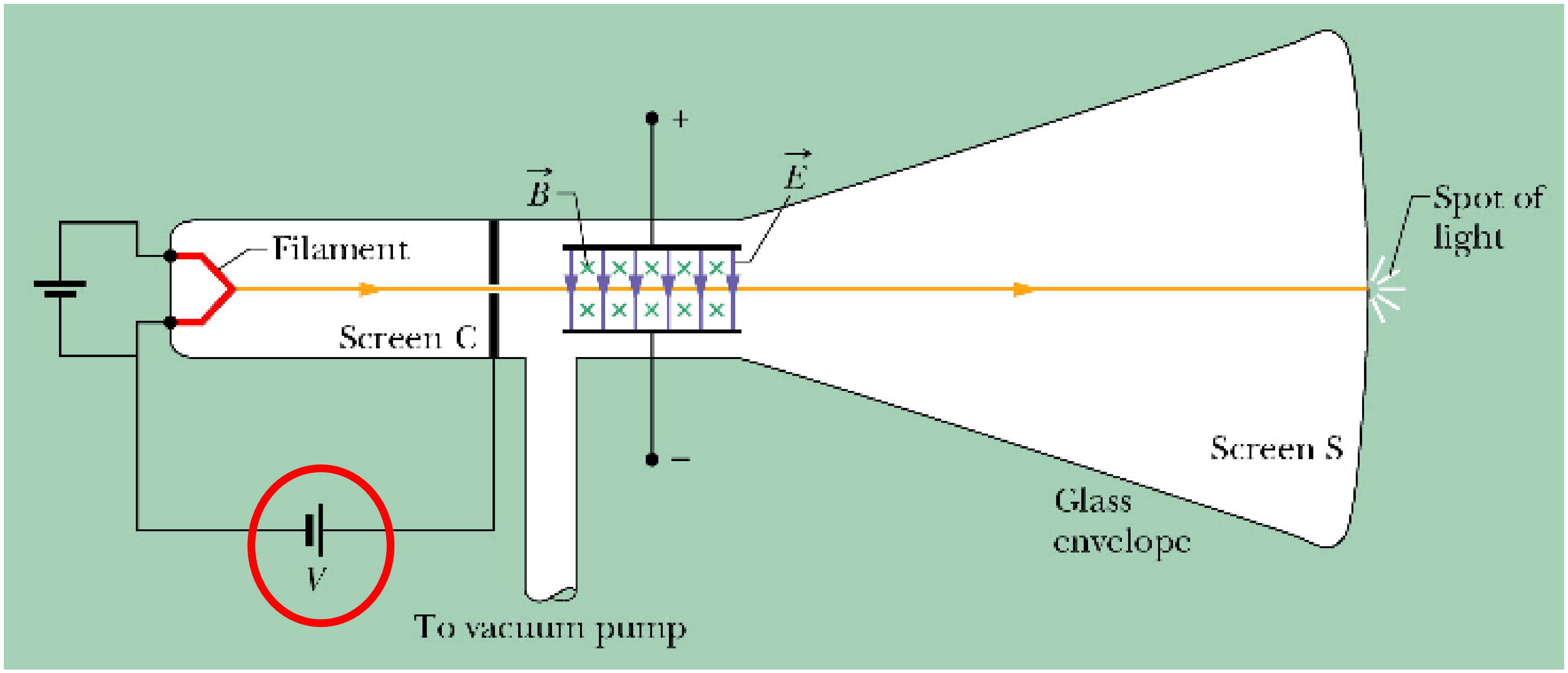
(b)

For a dipole, the shape of the equipotentials is more complicated:



(c)

Cathode Ray Tube



Electron gun: potential V gives electron energy in eV.

$$K = \frac{1}{2}mv^2 = qV$$

So if V = 500 volts, electron energy is K = 500 eV

Be careful with SI units

Suppose we want to calculate the speed of this electron. We must first convert the energy from eV to SI units.

$$K = \frac{1}{2}mv^2 = qV = 500\ eV$$

$$= (500\ eV) \times (1.6 \times 10^{-19}\ J/eV) = 8 \times 10^{-17}\ J$$

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2 \times 8 \times 10^{-17}\ J}{9.1 \times 10^{-31}\ kg}} = 1.3 \times 10^{7}\ m/s$$

High energies

- **For X-ray machines, accelerate electrons with potentials of thousands of volts, so we speak of kinetic energies in *keV*.** $10^3\ eV$
- **In nuclear physics, accelerators produce beams of particles with energies in *MeV*.** $10^6\ eV$
- **In elementary-particle physics, high-energy particles beams have energies measured in giga-volts: *GeV*.** $10^9\ eV$

Capacitance

Chapter 25 homework:

Questions 1, 10

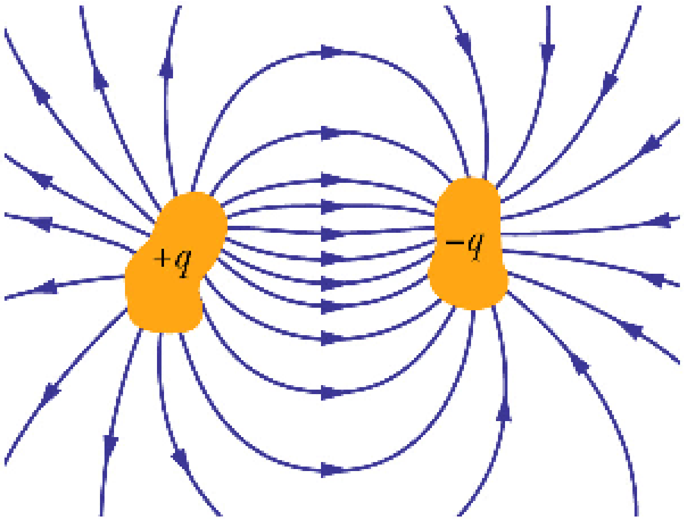
Problems 1, 9, 25, 39

Capacitors

- **Capacitors store charge and energy.**
- **Definition of Capacitance: C = Q/V**
- **Calculate stored energy:**

$$U = \frac{Q^2}{2C}$$

Two Conductors Form a Capacitor



$$\Delta V = \int_{-q}^{+q} \vec{E} \cdot d\vec{s}$$

Given the amount of charge q.

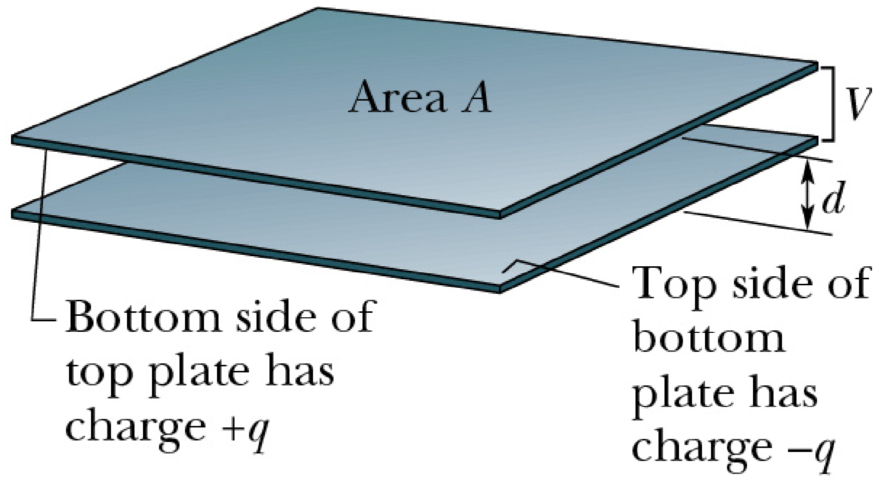
Let V be the potential difference ΔV.

Define the ***capacitance*** C = q/V.

SI unit is the farad: 1 F = 1C/V

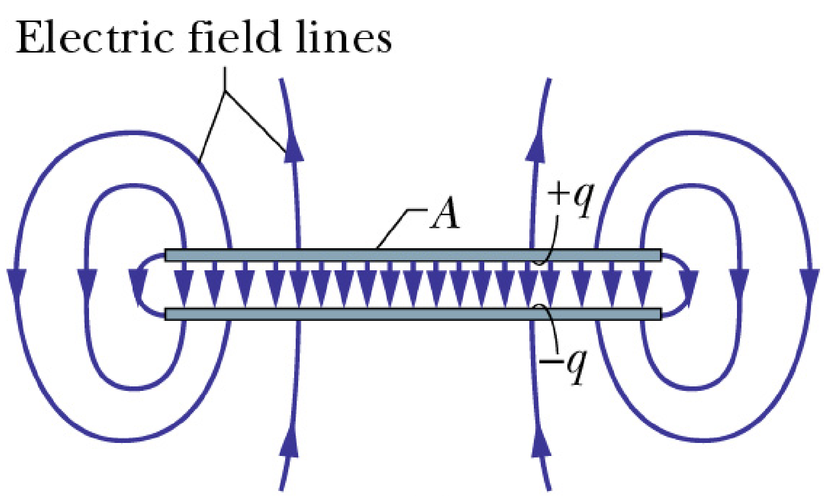
Case 1: The Parallel-plate Capacitor

The simplest capacitor is made of two large parallel metal plates very close together.



(a)

Because of its construction, we see that all the charge is on the inner surfaces.



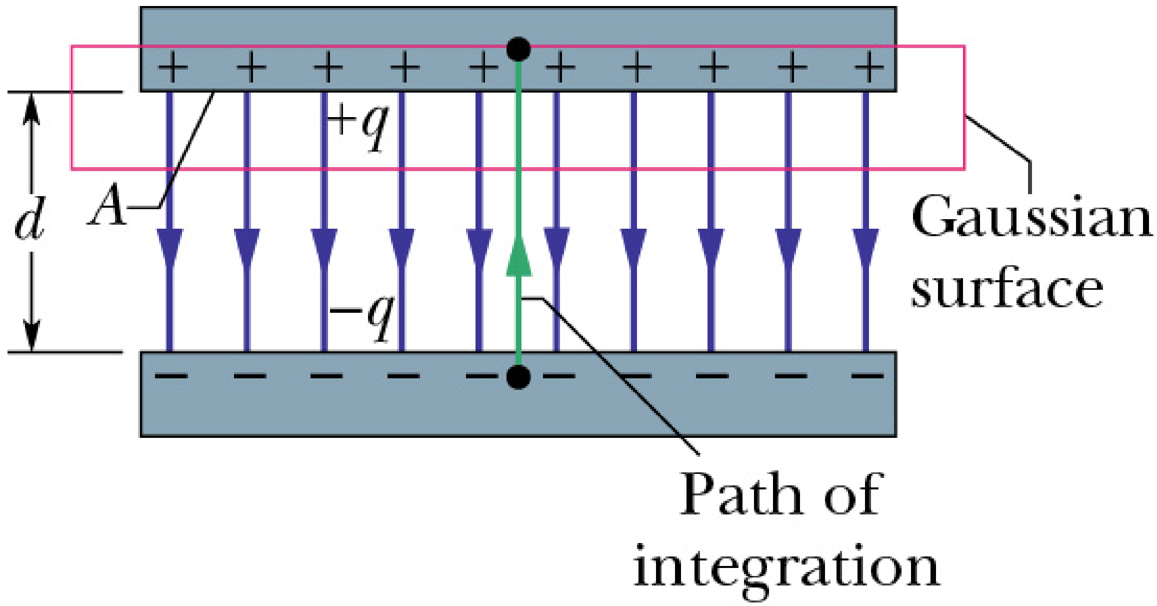
(b)

Parallel-Plate Capacitor

$$q = \sigma A$$

$$E = \sigma / \varepsilon_0$$

$$V = Ed$$



$$C = q / V = \frac{\sigma A}{\sigma d / \varepsilon_0} = \frac{\varepsilon_0 A}{d}$$

Potential Energy

- I must do work to charge up a capacitor.

- This energy is stored in the form of ***electric potential energy***.

- We will show that this is $\boldsymbol{U = \frac{Q^2}{2C}}$

- Then we will see that this energy is stored in the electric field, with a volume ***energy density***

$$\boxed{u = \frac{1}{2}\varepsilon_0 E^2} \qquad \left(J/m^3\right)$$

Derivation of Capacitor Energy

- If capacitor has charge q and I ***add*** charge dq, then I must do ***work*** $dW = Vdq$

(Remember the ***definition*** *of potential difference!)*

- For each dq I add, I must work harder because there is a stronger field against me. If I start with q=0 and end with q=Q, then my total work is

$$W = \int Vdq = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

Energy density of the electric field

- **Special case of parallel-plate capacitor:**

$$U = Q^2/2C = \tfrac{1}{2}CV^2 = \tfrac{1}{2}CE^2d^2$$

$$But \quad C = \varepsilon_0 A/d \quad and \quad Vol. = Ad$$

$$\therefore \quad U = \tfrac{1}{2}(\varepsilon_0 A/d)E^2d^2 = \tfrac{1}{2}\varepsilon_0 E^2 Ad$$

And so we get $$\boxed{\boldsymbol{u = (U)/(Vol.) = \tfrac{1}{2}\varepsilon_0 E^2}}$$

- Turns out to be true in general: an electric field ***always*** carries this energy density

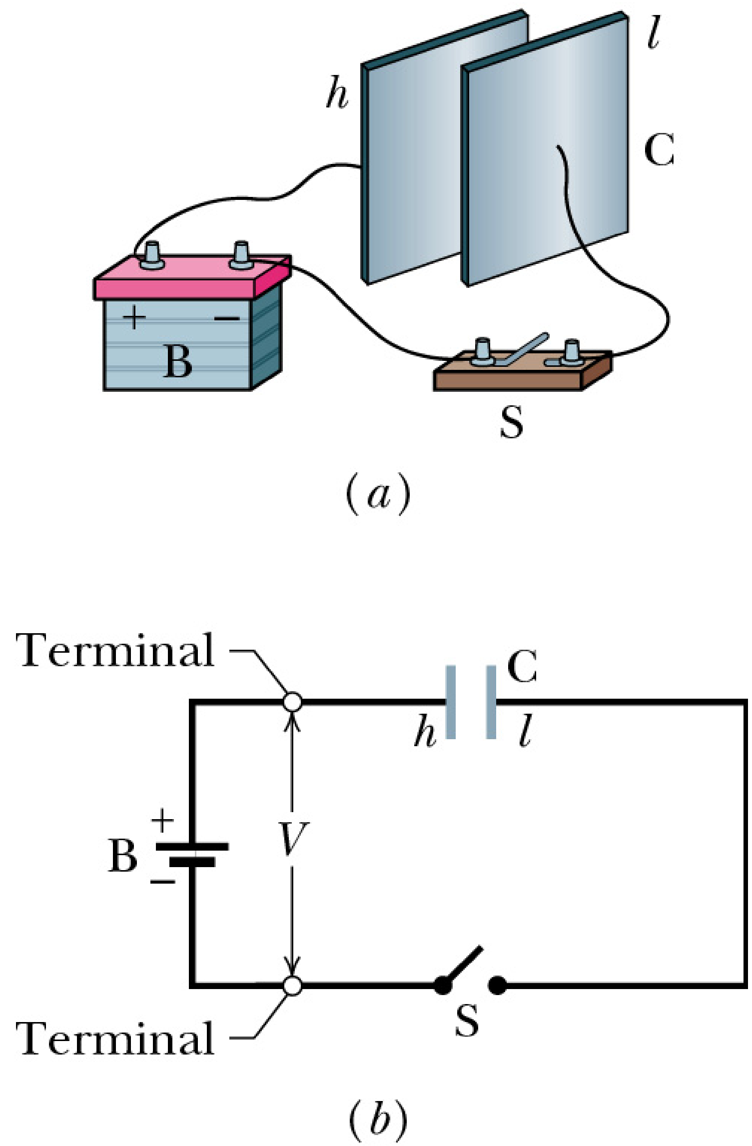
A Numerical Example

1. Suppose we have 2 metal plates, 20cm by 30cm, set parallel to each other, separated by a gap of 3 mm. What will be the capacitance of this setup?

$$C = \frac{\varepsilon_0 A}{d} = \frac{9 \times 10^{-12} \times 0.2 \times 0.3}{0.003}$$

$$C = 1.8 \times 10^{-10}\, F = 0.18\, nF = 180\, pF$$

2. Now suppose we connect a 12-volt battery between these two plates, so that the potential difference across this capacitor is V = 12 volts. **How much charge will be on the plates?**



2. If the potential difference is 12 volts how much charge will be on the plates?

Use the definition of capacitance:

$$C = q / V$$

$$\therefore \quad q = CV = (0.18\,nF) \times (12\,V) = 2.16\,nC$$

Thus the positive plate will have +2.16 nC and the negative plate will have –2.16 nC.

3. What is the strength of the electric field between the plates?

Method A: Remember V is work to move 1 C between plates, force times distance, so

$$V = Ed$$

$$E = V / d = 12V / .003m = 4000 V / m$$

Method B: From Gauss's Law E=σ/ε_0, so

$$\boldsymbol{E = (Q / A) / \varepsilon_0 = \frac{2.16 \times 10^{-9}}{.2 \times .3 \times 9 \times 10^{-12}} = 4000 V / m}$$

4. How much energy is stored?

$$U = Q^2 / 2C$$

$$U = \frac{(2.16\,nC)^2}{2 \times (180\,pF)} = \frac{4.67 \times 10^{-18}\ SI}{360 \times 10^{-12}\ SI} = 13\,nJ$$

5. What voltage would be required to store 1 joule of energy?

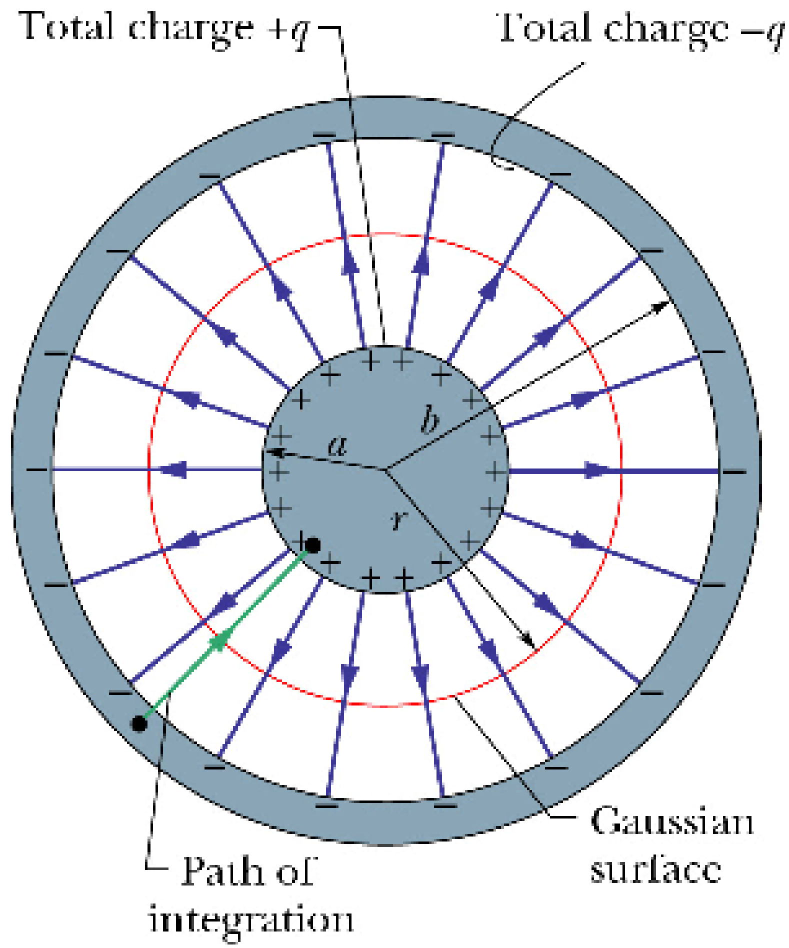
$$U = Q^2 / 2C \quad and \quad Q = CV$$

$$\therefore \quad U = \tfrac{1}{2} CV^2$$

$$So \quad V = \sqrt{2U / C} = \sqrt{2 \times 1 / 1.8 \times 10^{-10}}$$

$$= 1.1 \times 10^5\, V = 110\, kV$$

Case 2: The Spherical Capacitor



Use Gauss to find E(r):

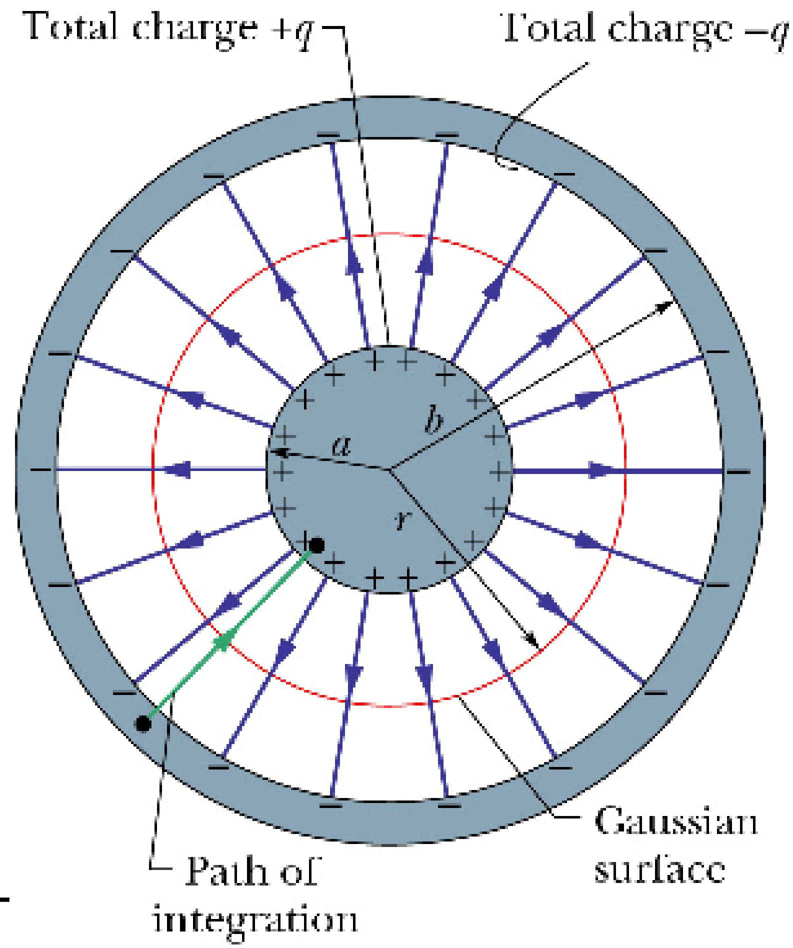
$$\Phi_{out} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(r) \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E(r) = \frac{q}{4\pi r^2 \varepsilon_0} = \frac{kq}{r^2}$$

$$V = \int_a^b E(r)\,dr$$

Spherical Capacitor Continued



$$V = \int E dr = \int_a^b \frac{kq}{r^2} dr$$

$$= kq\left(\frac{1}{a} - \frac{1}{b}\right) = kq\frac{b-a}{ab}$$

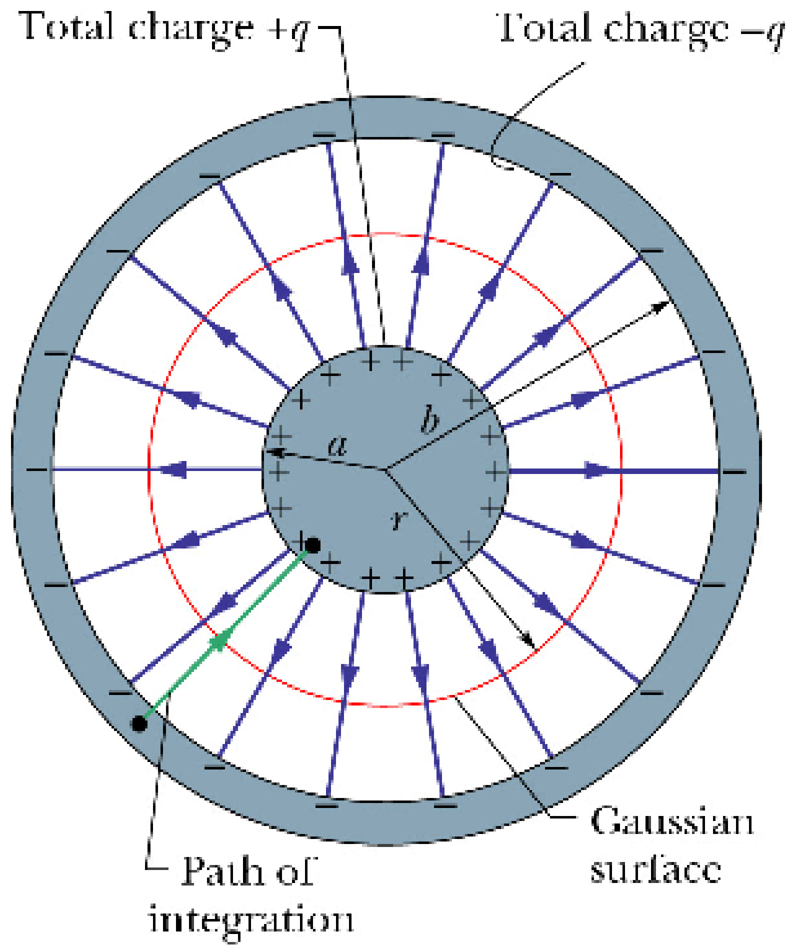
$$\therefore \quad C = q/V = \frac{ab/k}{b-a} = \frac{4\pi\varepsilon_0 ab}{b-a}$$

Case 3: The Cylindrical Capacitor

Application: capacitance per unit length of a coaxial cable.

$$V = \int_a^b E(r)\,dr$$

What is E(r) in this case?
Again use Gauss's Law.

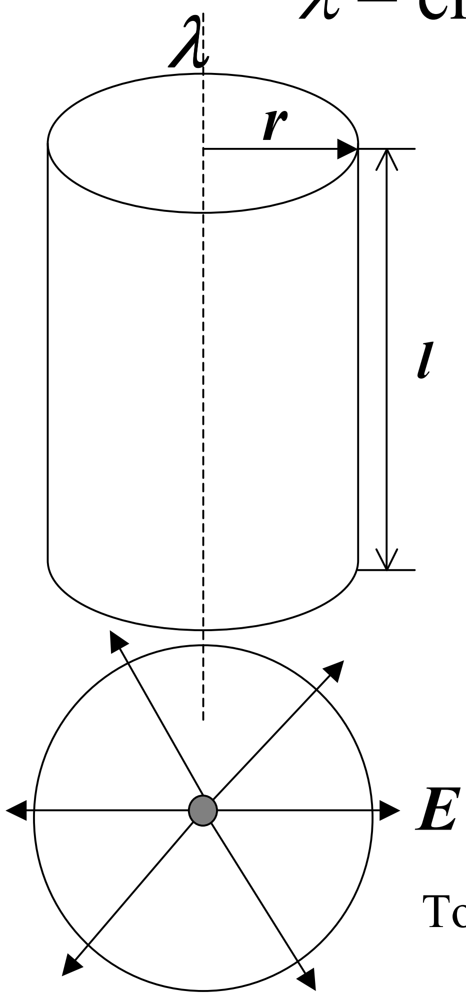


Cross section of cable

λ = **charge per unit length**

Review: Long Line of Charge

λ = charge/length = linear charge density



Top view

$$\Phi = \frac{Q}{\varepsilon_0}$$ **Gauss' law**

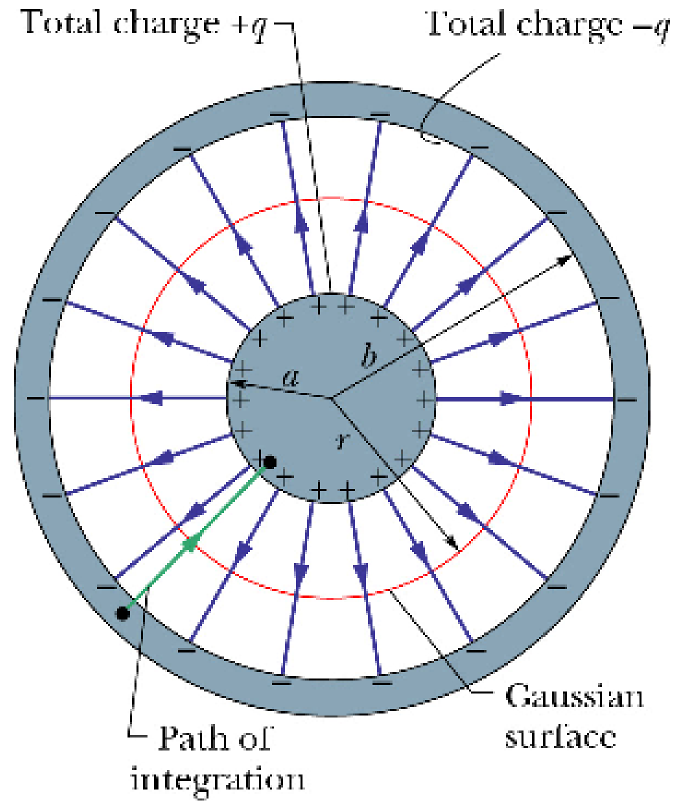
$$Q = l\lambda$$

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = EA$$

$$A = 2\pi r l$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Back to Cylindrical Capacitor

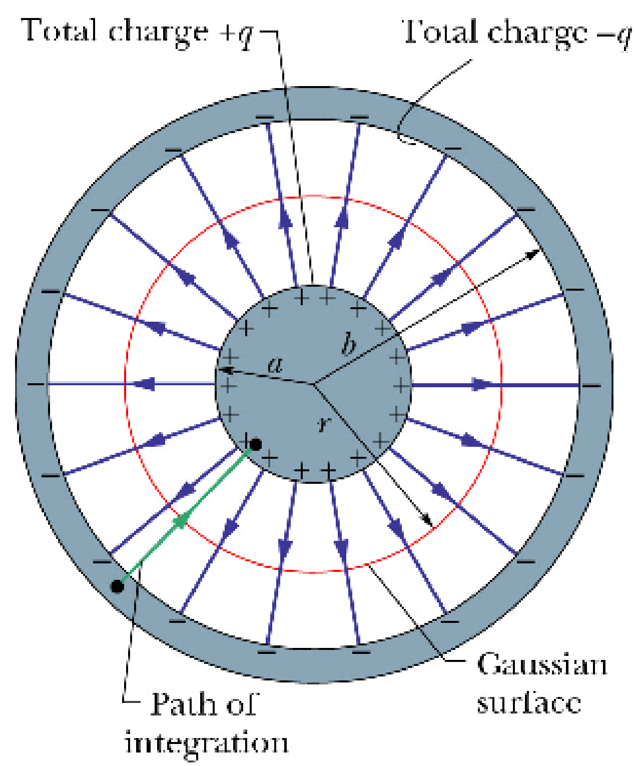


$$V = \int_a^b E(r)dr$$

$$= \int_a^b \frac{\lambda}{2\pi\varepsilon_0 r} dr = \frac{\lambda}{2\pi\varepsilon_0} \int_a^b \frac{dr}{r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \left[\ln r\right]_a^b = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{b}{a}$$

Finish Cylindrical Capacitor



Capacitance per unit length:

$$C = q / V$$

so $$\frac{C}{L} = \frac{\lambda}{V} = \frac{\lambda}{\dfrac{\lambda}{2\pi\varepsilon_0}\ln\dfrac{b}{a}}$$

$$= \frac{2\pi\varepsilon_0}{\ln(b/a)}$$