Chapter 23: Gauss's Law

Homework:

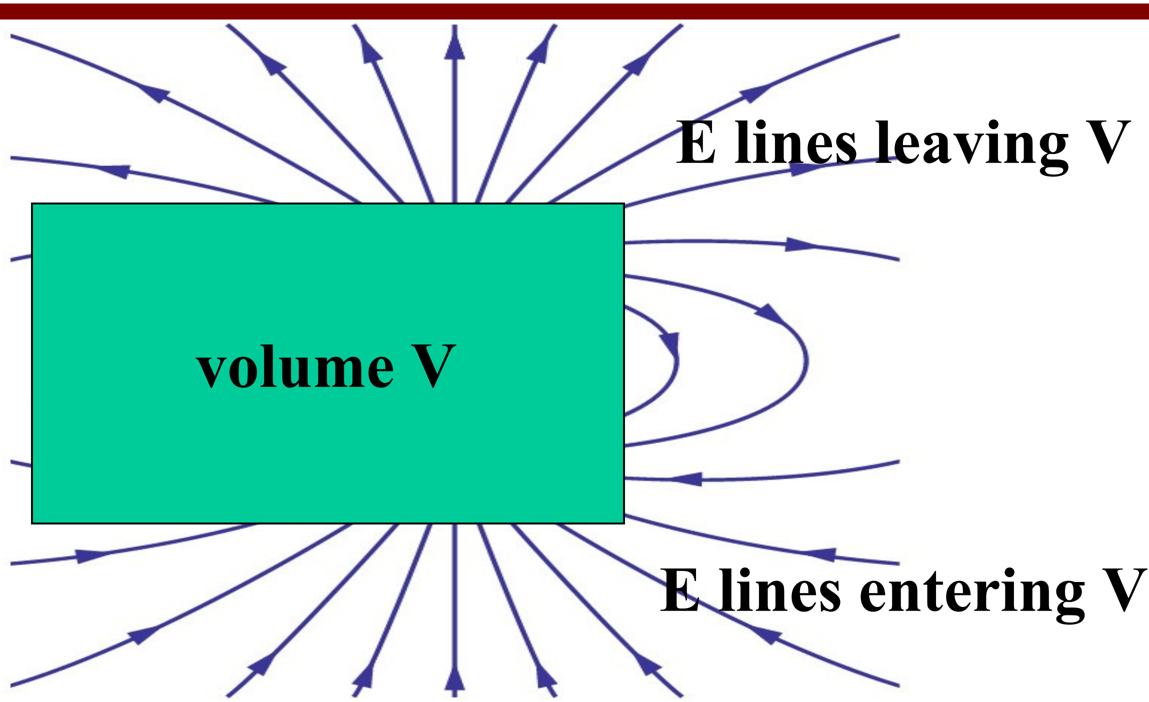
- Read Chapter 23
- Questions 2, 5, 10
- Problems 1, 5, 32

Gauss's Law

- **Gauss's Law is the first of the four *Maxwell Equations* which summarize all of electromagnetic theory.**
- **Gauss's Law gives us an *alternative* to Coulomb's Law for calculating the electric field due to a given distribution of charges.**

Gauss's Law: The General Idea

The net number of electric field lines which leave any volume of space is proportional to the net electric charge in that volume.



Flux

The ***flux Φ*** of the ***field E*** through the ***surface S*** is **defined** as

$$\Phi = \int_S \vec{E} \cdot d\vec{A}$$

The **meaning** of flux is just the **number of field lines** passing through the surface.

Best Statement of Gauss's Law

The outward flux of the electric field through any closed surface equals the net enclosed charge divided by ε_0.

Gauss's Law: The Equation

$$\oint_S \vec{E} \cdot d\vec{A} = Q_{enc} / \varepsilon_0$$

- S is any **closed** surface.
- Q_{enc} is the **net** charge **enclosed** within S.
- dA is an element of area on the surface of S.
- $\vec{dA}$ is in the direction of the **outward normal.**

$$\varepsilon_0 = 8.85 \times 10^{-12} \ SI\ units$$

Flux Examples

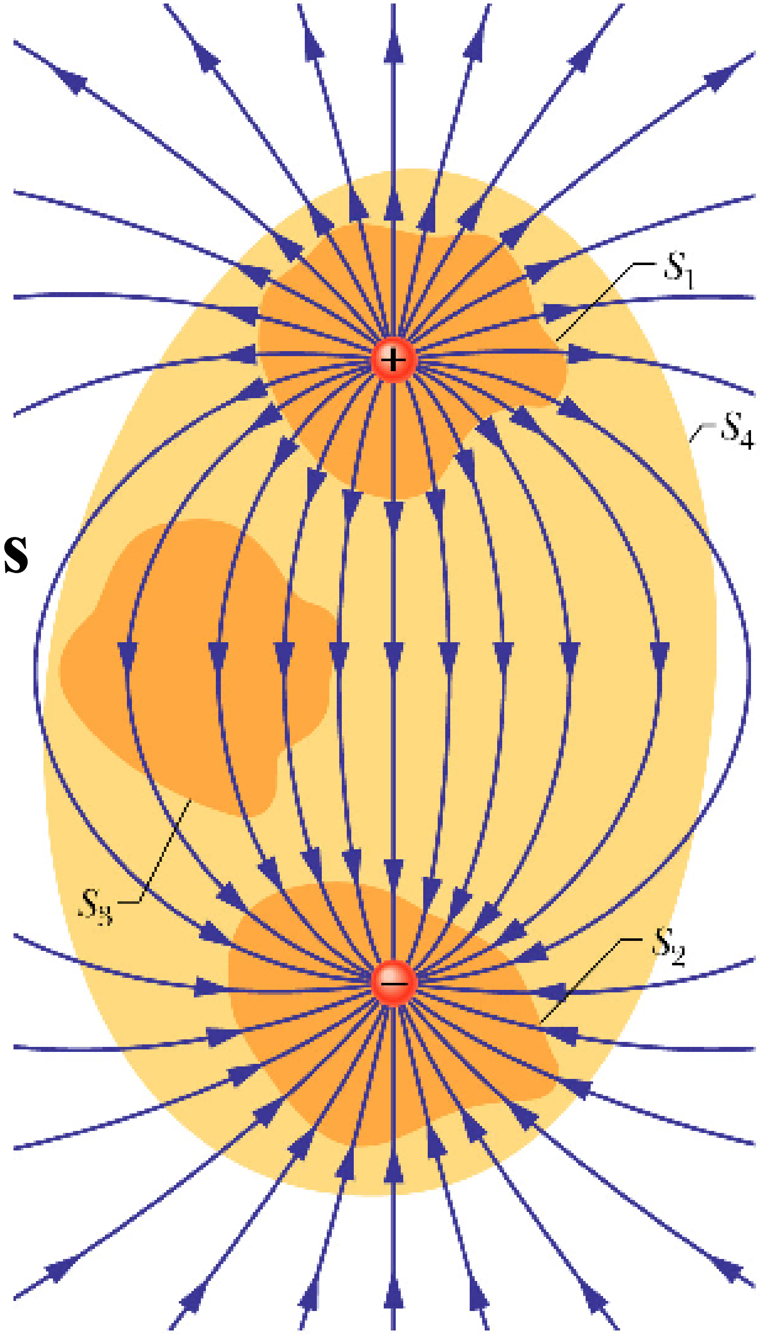
Assume two charges, +q and –q.
Find fluxes through surfaces.
Remember flux is negative if lines are entering closed surface.

$\Phi_1 = +q/\varepsilon_0$

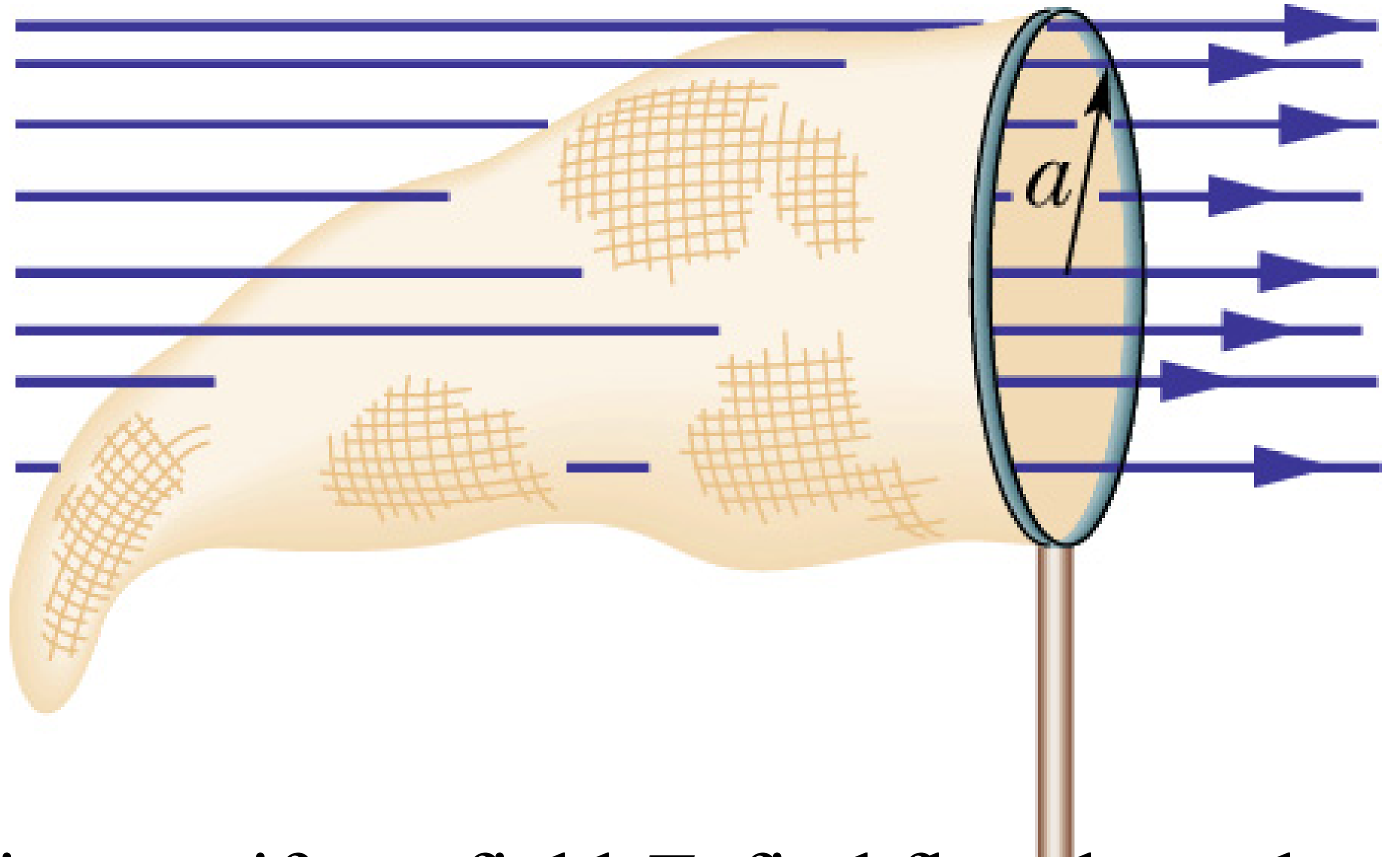
$\Phi_2 = -q/\varepsilon_0$

$\Phi_3 = 0$

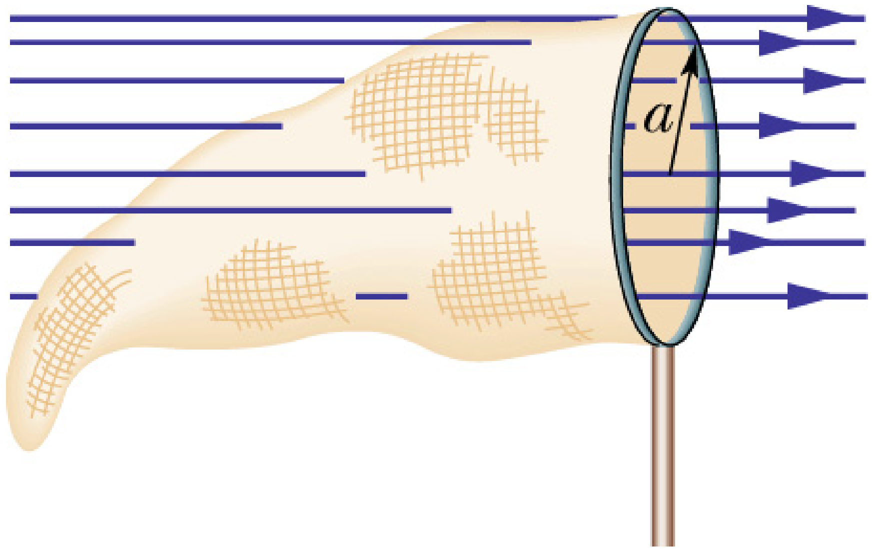
$\Phi_4 = (q - q)/\varepsilon_0 = 0$



Another Flux Example



Given uniform field ***E***, find flux through net.

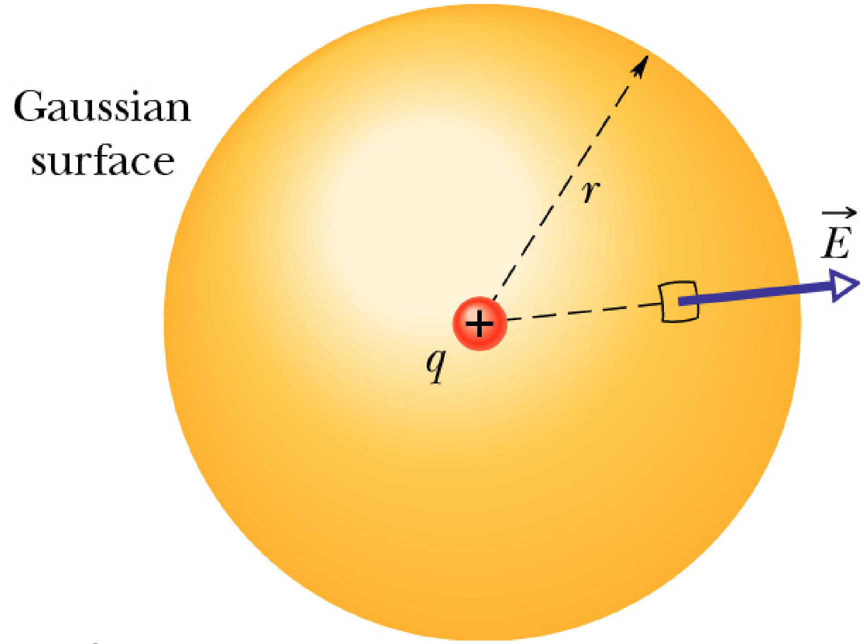


Must have ***closed surface,*** so let ***S*** be (net+circle).
Then $Q_{enc} = 0$ so Gauss's law says $\Phi_S = 0$.

$$But \quad \Phi_{circle} = \pi a^2 E$$

$$\therefore \quad \Phi_{net} = \underline{-\pi a^2 E}$$

Gauss ⇒ Coulomb



Given a point charge, draw a concentric sphere and apply Gauss's Law:

$$\vec{E} = E(r)\hat{r}$$

$$\Phi = \oint \vec{E} \cdot d\vec{A} = E(r)\, 4\pi\, r^2$$

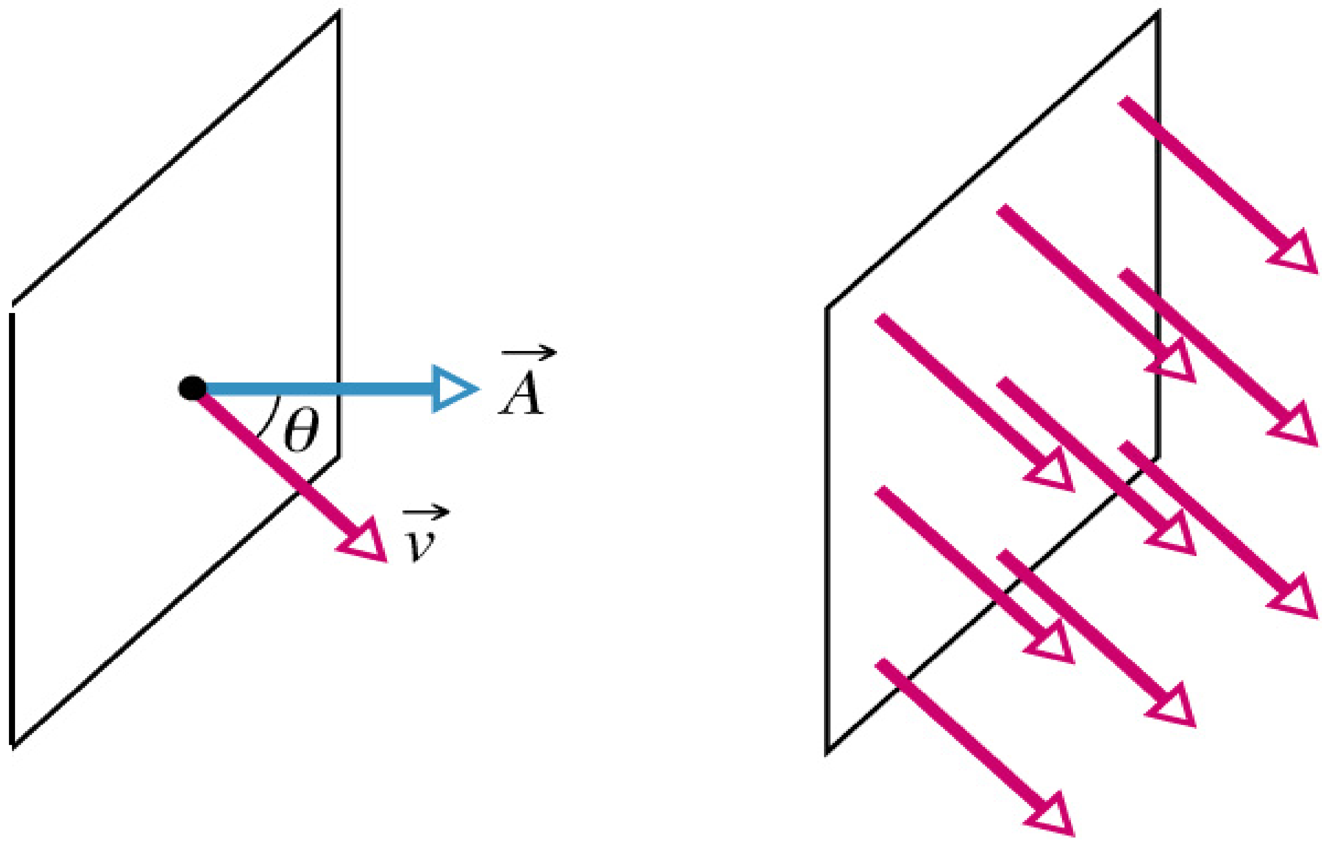
Gauss ⇒ $\Phi = q / \varepsilon_0$

$$\therefore \quad E(r) = q / (4\pi\, r^2 \varepsilon_0) = \boxed{kq / r^2} \; !$$

If a vector field v passes through an area A at an angle θ, what is the name for the product

$$\vec{v} \cdot \vec{A} = vA\cos\theta \quad ?$$

Q.23-1



(1) curl (2) energy (3) flux (4) gradient

Q.23-1 What is the name for the product

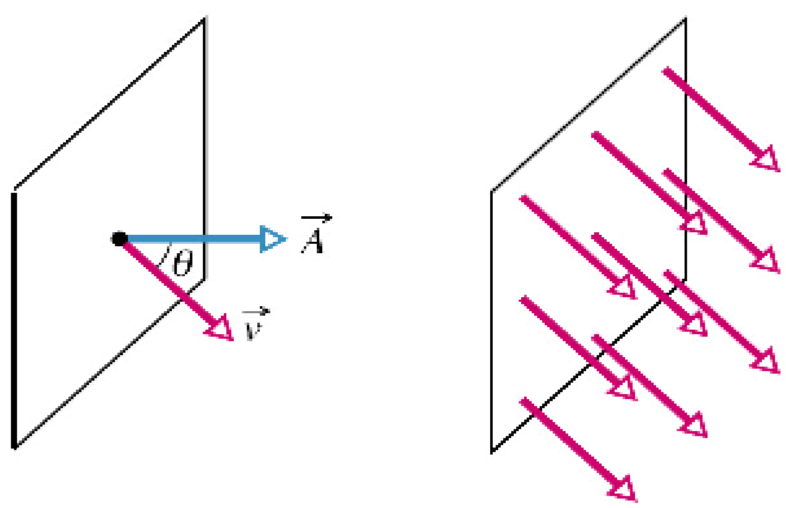
$$\vec{v} \cdot \vec{A} = vA\cos\theta \quad ?$$

1. Curl
2. Energy
3. Flux
4. Gradient

Q.23-1

$$\Phi = \vec{v} \cdot \vec{A} = vA\cos\theta$$

= ***flux*** from the Latin "to flow".



(1) curl (2) energy (3) flux (4) gradient

Q.23-2

Two charges Q_1 and Q_2 are inside a ***closed cubical box*** of side a. What is the ***net outward flux*** through the box?

(1) $\Phi = 0$

(2) $\Phi = \left(Q_1 + Q_2\right)/\,\varepsilon_0$

(3) $\Phi = k\left(Q_1 + Q_2\right)/\,a^2$

(4) $\Phi = \dfrac{Q_1 + Q_2}{4\pi\varepsilon_0 a^2}$

(5) $\Phi = \dfrac{Q_1 - Q_2}{4\pi\varepsilon_0 a^2}$

Q.23-2 **Two charges Q_1 and Q_2 are inside a *closed cubical box* of side a. What is the *net outward flux* through the box?**

1. $\Phi = 0$
2. $\Phi = (Q_1 + Q_2)/\varepsilon_0$
3. $\Phi = k(Q_1 + Q_2)/a^2$
4. $\Phi = \dfrac{Q_1 + Q_2}{4\pi\varepsilon_0 a^2}$
5. $\Phi = \dfrac{Q_1 - Q_2}{4\pi\varepsilon_0 a^2}$

Q.32-2

Two charges Q_1 and Q_2 are inside a *closed cubical box* of side *a*. What is the *net outward flux* through the box?

$$\Phi = 0$$

$$\boxed{\Phi = \left(Q_1 + Q_2\right)/\varepsilon_0}$$

$$\Phi = k\left(Q_1 + Q_2\right)/a^2$$

$$\Phi = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 a^2}$$

$$\Phi = \frac{Q_1 - Q_2}{4\pi\varepsilon_0 a^2}$$

Gauss:

The outward flux of the electric field through any closed surface equals the net enclosed charge divided by ε_0.

Application of Gauss's Law

- We want to compute the electric field at the surface of a charged metal object.
- This gives a good example of the application of Gauss's Law.
- First we establish some facts about good conductors.
- Then we can get a neat useful result:

$$E = \sigma / \varepsilon_0$$

Fields in Good Conductors

Fact: **In a steady state the electric field inside a good conductor must be zero.**

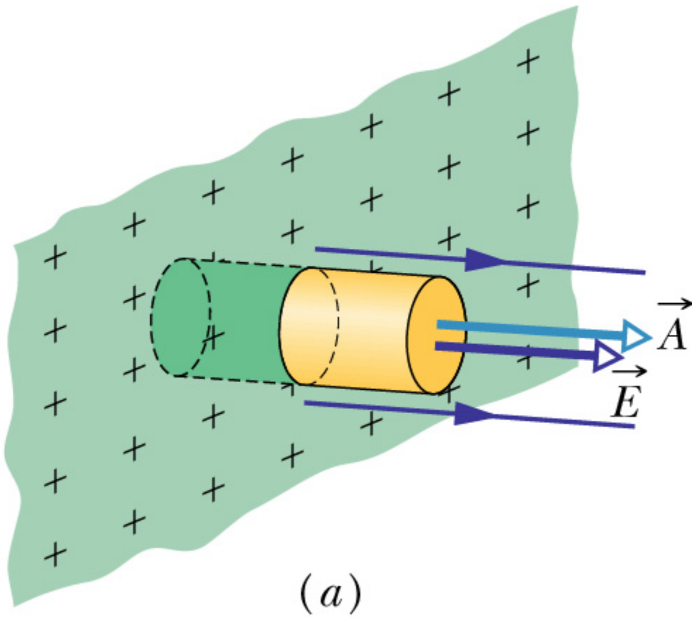
Why? **If there were a field, charges would move. Charges will move around until they find the arrangement that makes the electric field zero in the interior.**

Charges in Good Conductors

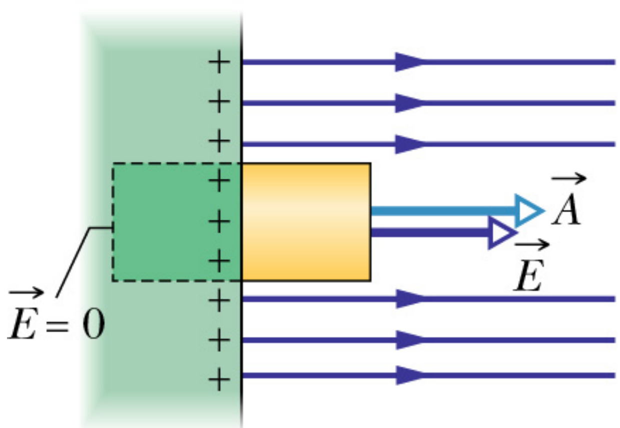
Fact: **In a steady state, any net charge on a good conductor must be entirely on the surface.**

Why? **If there were a charge in the interior, then by Gauss's Law there would be a field in the interior, which we know cannot be true.**

Field at the Surface of a Conductor



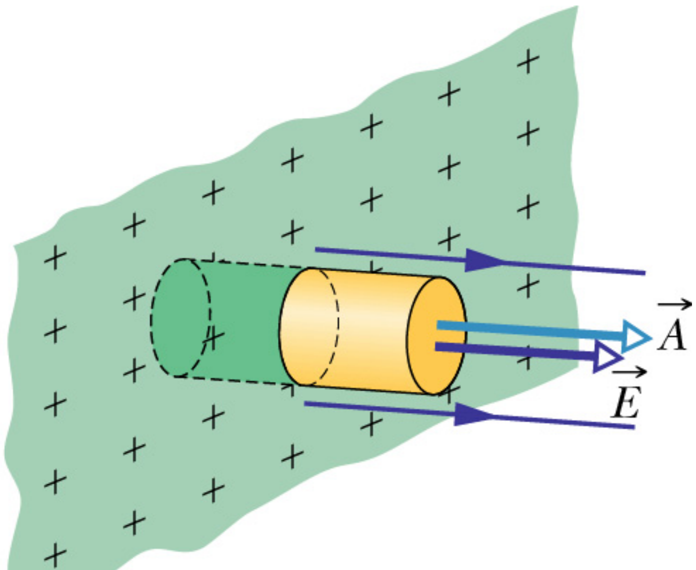
(a)



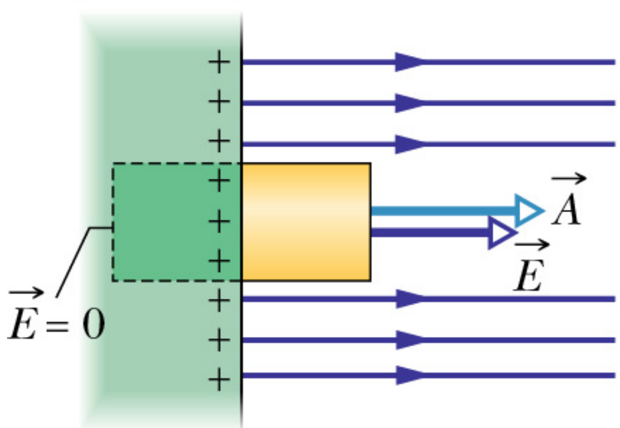
(b)

- Construct closed gaussian surface, sides perpendicular to metal surface, face area = A.
- Flux through left face is zero because E=0.
- Field is perpendicular to surface or charges would move, therefore flux through sides is 0.
- So net outward flux is EA.

Field at the Surface of a Conductor



(a)



(b)

- Let σ stand for the surface charge density (C/m^2)
- Then $Q_{enc} = \sigma A$.
- Now Gauss's Law gives us

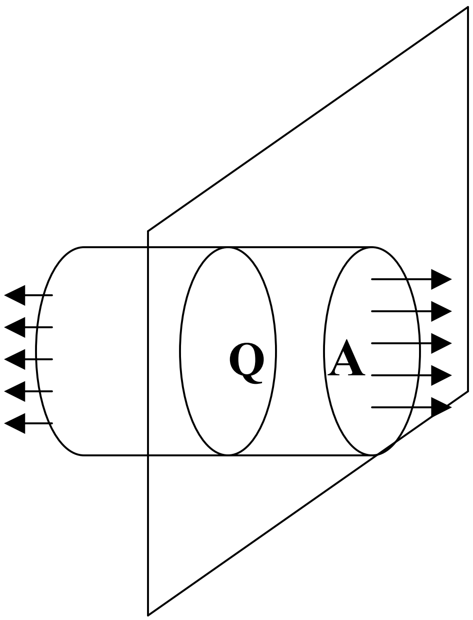
$$\Phi = Q_{enc} / \varepsilon_0$$

$$EA = \sigma A / \varepsilon_0$$

$$E = \sigma / \varepsilon_0$$

Large Sheet of Charge

σ = charge /area = surface charge density



$$\Phi = \frac{Q}{\varepsilon_0}$$ **Gauss' law**

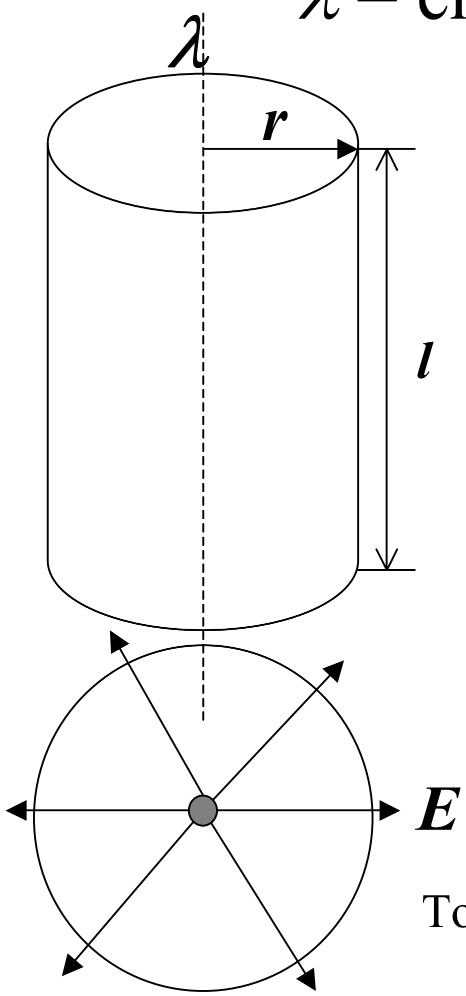
$$Q = \sigma A$$

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = 2EA$$

$$2EA = \frac{\sigma A}{\varepsilon_0} \qquad E = \frac{\sigma}{2\varepsilon_0}$$

Long Line of Charge

λ = charge/length = linear charge density



Top view

$$\Phi = \frac{Q}{\varepsilon_0}$$ **Gauss' law**

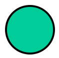
$$Q = l\lambda$$

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = EA$$

$$A = 2\pi r l$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Summary for different dimensions



Point charge, d=0

$$\Phi = 4\pi\, r^2 E = Q/\varepsilon_0$$

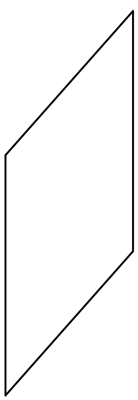
$$E = Q/(4\pi\varepsilon_0 r^2)$$

Line charge, d=1

$$\boxed{E \propto \frac{1}{r^{2-d}}}$$

$$\Phi = 2\pi\, rlE = \lambda\, l/\varepsilon_0$$

$$E = \lambda/(2\pi\varepsilon_0 r)$$



Surface charge, d=2

$$\Phi = 2AE = \sigma A/\varepsilon_0$$

$$E = \sigma/(2\varepsilon_0)$$

Potential

Outline for today

- **Potential as energy per unit charge.**
- **Third form of Coulomb's Law.**
- **Relations between field and potential.**

Potential Energy per Unit Charge

Just as the ***field*** is defined as ***force*** per unit charge, the ***potential*** is defined as ***potential energy*** per unit charge:

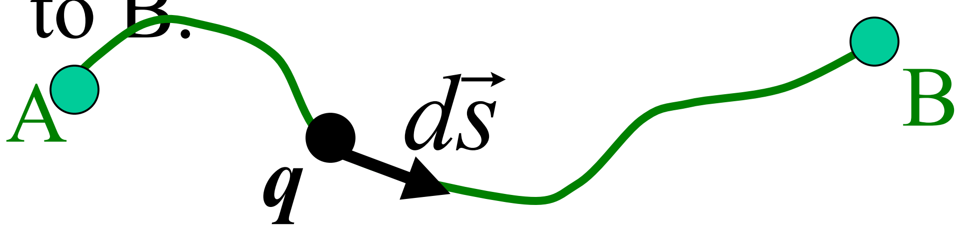
$$\vec{F} = q\vec{E} \quad and \quad U = qV$$

The SI unit for potential is the ***volt*.** (1V=1J/C)

- Potential is often casually called "voltage".
- As with potential energy, it is really the ***potential difference*** which is important.

Potential Energy Difference

If a charge q is originally at point A, and we then move it to point B, the ***potential energy*** will increase by the amount of work we have done in carrying the charge from A to B.



$$dW = \vec{F}_{ext} \cdot d\vec{s}$$

$$U_B - U_A = W_{AB} = \int_A^B \vec{F}_{ext} \cdot d\vec{s}$$

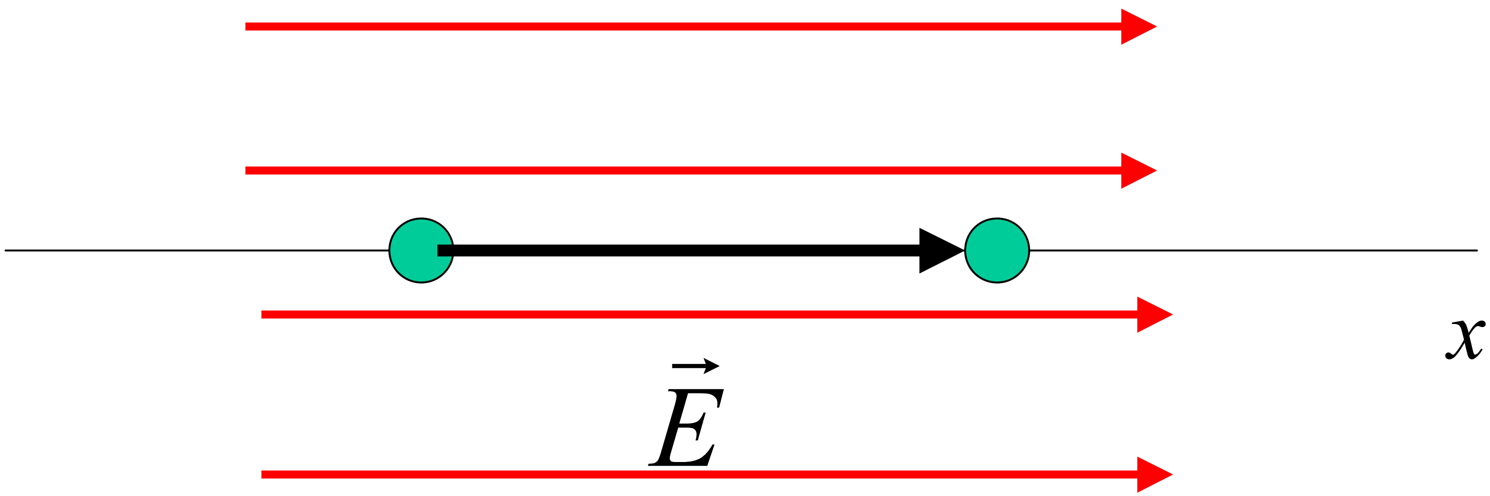
Potential Difference

The ***potential difference*** between point A and point B is this work ***per unit charge.***

$$W_{AB} = qV_{AB} = \int_A^B \vec{F}_{ext} \cdot d\vec{s} = -\int_A^B q\vec{E} \cdot d\vec{s}$$

$$\therefore \quad V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

Relation between E and V



$$\Delta V = -\int E_x dx$$

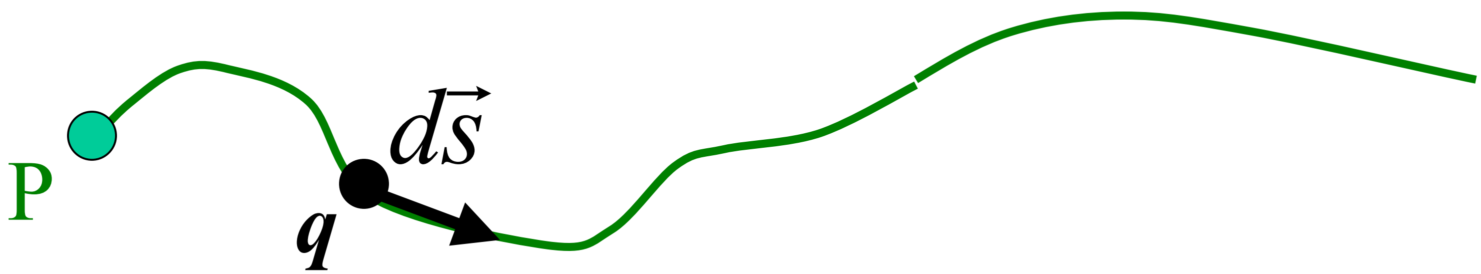
$$\boldsymbol{And} \quad \boldsymbol{E_x = -\frac{dV}{dx}} \quad \boldsymbol{etc.}$$

Potential Relative to Infinity

- **We have defined potential difference ΔV. But to have a value for V itself we need to decide on a zero point.**
- **In circuits, we define the *earth* to have V=0, and often *ground* the circuit.**
- **In electrostatics we normally define V=0 far away from all charges. Then at point P we write V(P) to mean V(P) - V(∞).**

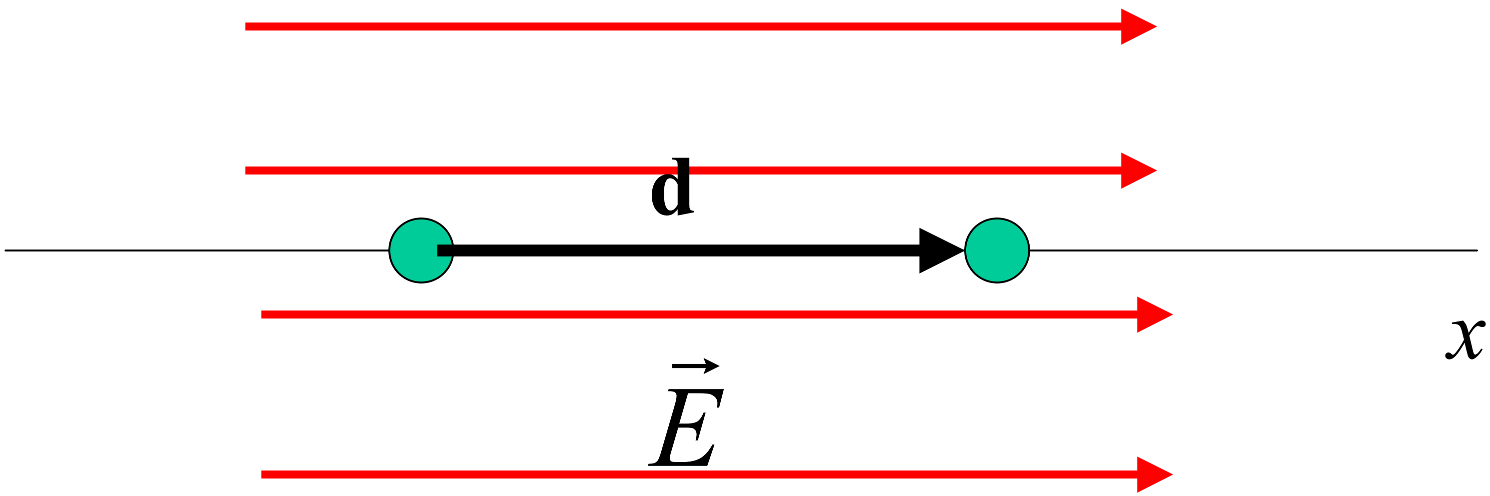
Potential at a point

The potential at point P is the work required to bring a one-coulomb test charge from far away to the point P.



$$V(P)=\int_{\infty}^{P}\frac{\vec{F}\cdot d\vec{s}}{q}=\int_{P}^{\infty}\vec{E}\cdot d\vec{s}$$

Example 1: Uniform Field



$$\Delta V = -\int E dx = -Ed$$

$$V(0) - V(d) = \underline{Ed}$$

Work required to push a 1-coulomb test charge against the field from x=d to x=0.

"Work = force X distance"

Q.24-1

Suppose, using an xyz coordinate system, in some region of space, we find the electric potential is

$$V(x) = Ax^2$$ **where A is a constant.**

What is the x-component of the electric field in this region?

A. $E_x = Ax$

B. $E_x = Ax^2$

C. $E_x = -2Ax$

Q. 24-1 Given $V(x) = Ax^2$

where A is has the constant value $A = 25\,V/m^2$.

What is the x-component of the electric field?

Solution:

$$E_x = -\frac{d}{dx}V = -A\frac{dx^2}{dx} = -2Ax$$

(1) $E_x = Ax$ (2) $E_x = Ax^2$ $\boxed{(3)\ E_x = -2Ax}$

Furthermore

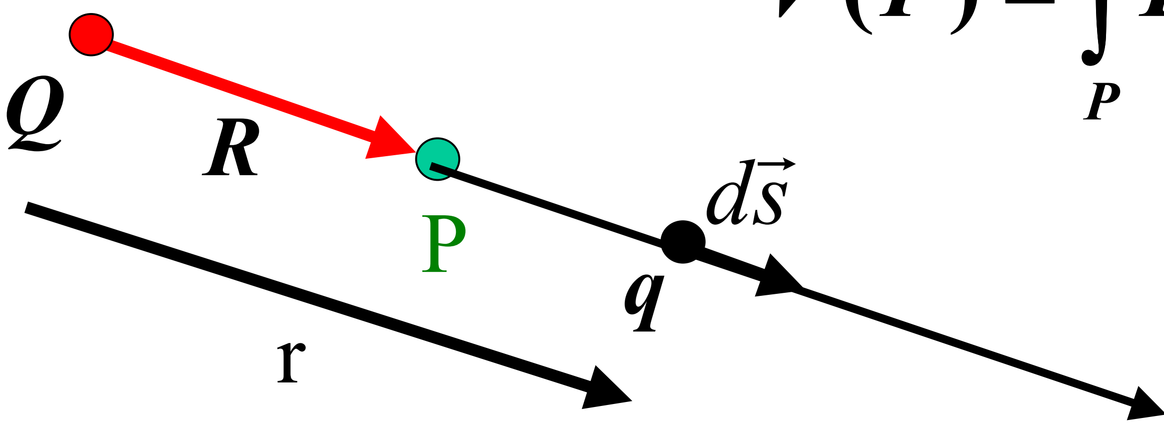
What are the y and z components of the electric field in the previous question?

$$E_y = -\frac{d}{dy}V = -A\frac{dx^2}{dy} = 0$$

and also $$E_z = -\frac{d}{dz}V = 0$$

Case of a Single Point Charge

$$V(P) = \int_P^\infty \vec{E} \cdot d\vec{s} = \int_R^\infty E(r)dr$$



$$V(R) = \int_R^\infty E(r)dr = kQ\int_R^\infty \frac{dr}{r^2}$$

$$= \left[-\frac{kQ}{r}\right]_R^\infty = -kQ\left[\frac{1}{\infty} - \frac{1}{R}\right] = \frac{kQ}{R}$$

Coulomb's Law for V

So we now have a ***third form of Coulomb's Law:***

1. $F = kQq / r^2$
2. $E = kQ / r^2$
3. $V = kQ / r$

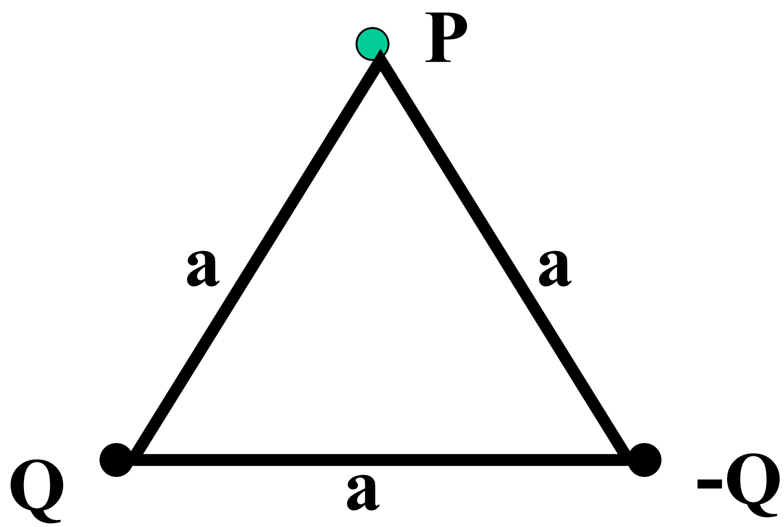
Potential is not a Vector

Adding forces and fields means ***adding vectors:*** finding the ***resultant vector.***

Adding potentials means ***adding numbers,*** and taking account of their signs. But it is much simpler than adding vectors.

Thus the third form of Coulomb's Law is the simplest!

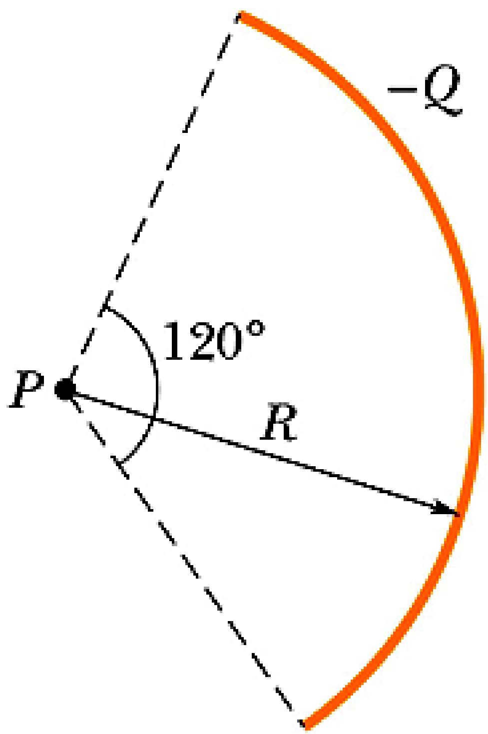
Example 1: Adding Potentials



$$V(P) = V_1 + V_2 = \frac{kQ}{a} + \left(-\frac{kQ}{a}\right) = 0$$

Note that the ***field*** at point P is ***not*** zero!

Q. 24-2



Uniformly charged rod with charge of –Q bent into arc of 120° with radius R.

***What is V(P), the electric potential at the center*?**

$$(1)\ +\frac{kQ}{R} \qquad (2)\ -\frac{kQ}{R} \qquad (3)\ \frac{kQ}{2R} \qquad (4)\ -\frac{kQ}{2R}$$

Q.24-2 **Uniformly charged rod with charge of –Q bent into arc of 120° with radius R.**

What is V(P), the electric potential at the center?

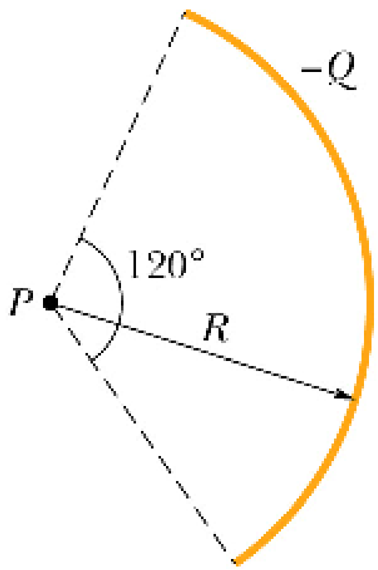
A. +kQ/R

B. -kQ/R

C. +kQ/2R

D. -kQ/2R

Q. 24-2



What is V(P), the electric potential at the center?

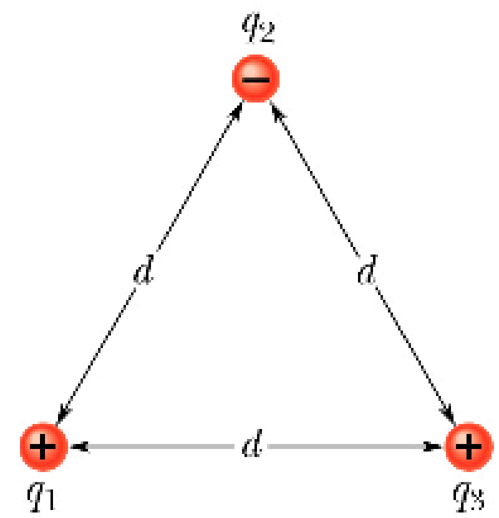
Solution: All bits of charge are at the same distance from P. Thus

$$V=\int\frac{kdq}{r}=\frac{k}{R}\int dq=\frac{k}{R}(-Q)$$

$(1)+\dfrac{kQ}{R}$ $\quad\boxed{(2)-\dfrac{kQ}{R}}$ $\quad(3)\dfrac{kQ}{2R}$ $\quad(4)-\dfrac{kQ}{2R}$

Potential Energy of Some Charges

The potential energy U of a group of charges is the work W required to assemble the group, bringing each charge in from infinity.



We can show that the result is

$$U = U_{12} + U_{13} + U_{23} + \cdots$$

Where the potential energy of each ***pair*** is of the form

$$\underline{U_{12} = kq_1q_2 / r_{12}}$$

Binding Energy

If the total potential energy ***U*** of a group of charges is ***negative*** that means we have to do work to pull them apart. The magnitude of this negative potential energy is called the ***binding energy***.

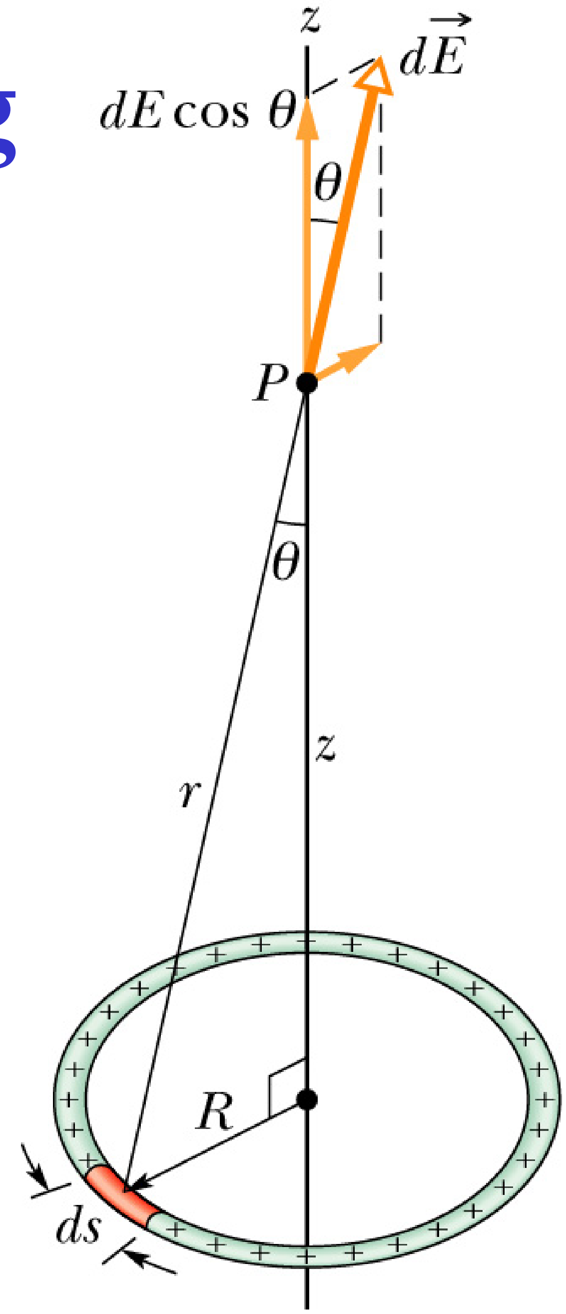
Examples:

- Removing an electron from an atom to form a positive ion.
- Removing a space probe from earth's gravitational field.

Example 3: Charged Ring

- In Ch. 22 the E field of a charged ring is calculated.
- Here we compute V first and then use it to get E.
- Get V at P on axis of circular ring, a distance z from center:

$$V(P) = \int \frac{kdQ}{r}$$



Example 3 Continued

Key point: All bits of charge are the same distance from point P!

$$V(P) = \int \frac{k\,dQ}{r} = \frac{k}{r}\int dQ = \frac{kQ}{\sqrt{R^2 + z^2}}$$

Note we have no struggling with angles or adding many little vectors as we do if we compute *E*.

Using V to get E

Now that we have V on the axis we can get E_z on the axis by differentiation:

$$E_z = -\frac{dV}{dz}$$

This is slightly messy, but we just need to remember that

$$\frac{df^n}{dz} = nf^{n-1}\frac{df}{dz}$$

where here

$$f^n = (R^2 + z^2)^{-1/2}$$

So finally we get

$$E(z) = -\frac{d}{dz}\frac{kQ}{\sqrt{R^2+z^2}} = \frac{kQz}{(R^2+z^2)^{3/2}}$$

which is exactly the result the textbook gets in Ch. 22 by direct integration of the field.

And also:

$$E_y = -\frac{d}{dy}V = -A\frac{dx^2}{dy} = 0$$

$$E_z = -\frac{d}{dz}V = 0$$

Furthermore

What are the y and z components of the electric field in the previous question?

$$E_y = -\frac{d}{dy}V = -A\frac{dx^2}{dy} = 0$$

and also $$E_z = -\frac{d}{dz}V = 0$$

Summary of Basics

- **$U = qV$**
- **$V = kQ/r$**
- **$W_{AB} = q\,(V_B - V_A)$**

$$\Delta V = -\int E_x dx$$

$$E_x = -\frac{dV}{dx} \quad etc.$$