Criteria for stability in bistable electrical devices with S- or Z-shaped current voltage characteristic

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Electronic devices exhibiting bistability in the current-voltage characteristics are considered from a unified viewpoint. We obtain simple relations for the stability of the different branches in the current-voltage characteristics. Criteria for oscillatory instabilities are discussed, and special conclusions for elements with S- or Z-shaped characteristics are drawn. The stabilization of the middle branch of the double-barrier resonant-tunneling diode in a circuit with effectively negative capacitance and negative resistance is derived in a simple way. © 1995 American Institute of Physics.

I. INTRODUCTION

Many semiconductor devices exhibit bistability between two conducting states at a given sample voltage. This results in different shapes of the current-voltage characteristics and may be due to quite different electrical transport mechanisms. It is the purpose of this work to demonstrate that this apparently very different individual behavior can be understood within a unified framework in terms of a few, very simple principles without recurrence to the underlying microscopic physics. Our intention is to provide the applied scientist who finds complicated multivalued current-voltage characteristics in some semiconductor device with natural criteria to identify the stability of different branches.

The bistability can be manifested in different types of the current-voltage characteristics as shown in Fig. 1. Here the full lines give branches which are usually stable, while the dashed lines are unstable under voltage controlled conditions. The most prominent type is the S-type characteristic as shown in Fig. 1(a), which occurs, e.g., in the regime of impact-ionisation breakdown, pnpn-diodes, the heterostructure hot-electron diode (HHED), and quantum-dot structures. The Z-shaped case [Fig. 1(b)] is currently widely discussed for the double-barrier resonant-tunneling diode (DBRTD, see, e.g., Ref. 8). It also appears in the post-breakdown regime of p-germanium. In both of these systems the stable branches of the loop-shaped case [Fig. 1(c)] have been observed for different conditions, too. An unconnected characteristic as shown in Fig. 1(d) occurs in the case of magnetococonductivity in n-Si and has been predicted for real-space transfer transistors. The current-voltage characteristics of doped superlattices even exhibit multistability between four and more different conducting states for the same operating conditions, as shown experimentally and theoretically in Ref. 13. These examples clearly state the physical relevance of different types of bistability for various devices.

All of these devices show at least two stable states with different conductivity for a given voltage in the range \( V_t < V < V_r \). Thus, there must exist an internal degree of freedom which determines in which state the device is operated. This internal degree of freedom has to be identified with an internal physical quantity such as the charge density in a quantum well, the electron temperature, the occupation of traps or impurities, the density of free carriers, or combinations of several physical quantities. Of course the nature of this physical quantity depends strongly on the semiconductor element considered.

In the following we assume that this internal degree of freedom can be represented by a single physical quantity which we will denote by \( a \) (e.g., \( a \) would be the electron temperature, if heating effects essentially determine the bistability). This means that all further degrees of freedom are not essential for the bistability and can be eliminated adiabatically for given \( a \) and \( U \). This is called the slaving principle in terms of nonlinear dynamics. This elimination is not possible for the quantity \( a \), as otherwise different stationary values would not be possible for a given voltage \( U \). The current density in the sample is then given by some equation

\[ j = j(a, U), \]

which has to be derived from a transport model for the individual device considered. As \( a \) is an independent dynamical variable, it is governed by some dynamic equation

\[ \frac{da(t)}{dt} = f(a, U) \]

resulting from the internal mechanism leading to the instability.

For instance, in case of several heterostructure devices \( a \) can be identified as the charge density in a quantum well. In this case \( f \) is simply given by the difference of the incoming and outgoing current. For example, such a microscopic modelling has been done for the DBRTD, the HHED, and a quantum-dot structure. Here it is essential to include Poisson's equation to treat the self-consistency with the electric potential. Similar treatments with different quantities \( a \) and different types of \( f(a, U) \) have been performed for other elements like pnpn-diodes and the low-temperature impact-ionization breakdown. (In the last case \( a \) is connected to the free carrier density and \( f(a, U) \) to the generation-recombination dynamics.)
FIG. 1. Different types of multistable current-density voltage characteristics which can be S-shaped (a), Z-shaped (b), loop-shaped (c), or disconnected (d). The full lines are usually stable under voltage controlled conditions, while the dotted lines can only be stabilized under special conditions.

In contrast to these works which are directed to single specific elements, here we want to give some insight into the generic features associated with bistability within a unified point of view. Thus, we do not specify the quantity \( a \) but discuss the influence of the dynamics of \( a(t) \) given by Eq. (2) upon the global behavior of the device in general terms. There are of course special situations which cannot be described by a single internal variable and which have to be regarded separately. Nevertheless, we think that most physical devices can be (and some have already been) described within such an approach.

II. STABILITY AT FIXED VOLTAGE

For a given voltage \( U \) the stationary states \( a^* \) of the internal quantity \( a \) are given by the relation \( f(a^*, U) = 0 \). In the case of multistability we have several stationary states \( a_1^* < a_2^* < a_3^* < ... \) in the range \( U_1 < U < U_r \). For simplicity we restrict ourselves to 3 stationary states in the following, as shown in Fig. 2. Cases where multistability between more than two stable states occurs can be treated analogously.

The curve \( f(a, U) = 0 \) separates the regions with \( f(a, U) > 0 \) where \( a(t) \) increases in time from regions with \( f(a, U) < 0 \) where \( a(t) \) decreases. As \( |a| \) should not grow unlimited in time for large \( |a| \) the function \( f(a, U) \) must be positive or negative for small or large \( a \), respectively, as depicted in Fig. 2. Regarding small perturbations, the stationary state \( a^* \) is stable if \( \frac{\partial f(a_2^*, U)}{\partial a} < 0 \). Here we use the notation

\[
\frac{\partial f(a_2^*, U)}{\partial a} \bigg|_{a=a^*}
\]

As \( f(a, U) \) changes its sign from plus to minus at \( a = a_2^* \) for fixed \( U \) with increasing \( a \), we find \( \frac{\partial f(a_2^*, U)}{\partial a} < 0 \). Therefore \( a_2^* \) is a stable state. Then \( f(a, U) \) must change its sign from minus to plus at the next zero \( a_3^* \). Thus, \( \frac{\partial f(a_3^*, U)}{\partial a} > 0 \) and \( a_3^* \) is an unstable fixed point under voltage controlled conditions. Similarly, \( a_4^* \) will be another stable fixed point. Therefore, we conclude that there must be at least three stationary states if two stable states are observed under voltage controlled conditions (i.e., if bistability occurs).

Now we want to consider the behavior of the solutions \( a_1^* \) as functions of \( U \). For \( U < U_1 \) we have only one stable solution \( a^* \). As can be seen from Fig. 1 this solution must develop continuously into one of the stable states in the bistable range if \( U \) is increased. Let this stable solution be \( a_1^* \). (If this were larger than the second stable solution, one can transform \( a \rightarrow -a \), so that \( a_1^* < a_2^* \) is again fulfilled.) Regarding the situation in Figs. 1(a),(b),(c) we find that only the solution \( a_1^* \) remains for \( U > U_r \), and we obtain a multi-valued relation \( a^*(U) \) which has the shape presented in Fig. 2(a). For the situation shown in Fig. 1(d) the shape of \( a^*(U) \) is plotted in Fig. 2(b).

If we now use the relation \( j = j[f(a^*(U), U)] \) from Eq. (1), we obtain the original current-density voltage characteristic from Fig. 1 with the additional unstable branch (dotted line). If we have the situation shown in Fig. 2(a) then the characteristic from Figs. 1(a) or (b) is obtained if \( j \) is monotonically increasing or decreasing in \( a \), respectively. If \( j \) is not monotonic in \( a \) the characteristic from Fig. 1(c) may occur. Of course, more complex characteristics are possible, too. Due to the stability regarding fluctuations in \( a \) the branches of the characteristic are stable (full line) or unstable (dotted line) for voltage controlled conditions as depicted in Fig. 1.

III. OPERATION VIA A LOAD RESISTANCE

In the following we consider the important case that the voltage \( U \) across the sample is not fixed, but the device is operated in a circuit as shown in Fig. 3. The capacitance \( C \) parallel to the sample is given by the sum of the device capacitance, the external capacitance and parasitic wire capacitances. Note that the resistance \( R \) usually is thought to be an external resistance, but it can also be the linear resistance of a part of the sample which is not associated with the bistable behavior (e.g., a contact resistance).

Now the temporal behavior of \( U(t) \) is determined by the circuit
FIG. 3. The bistable element embedded in a circuit with load resistor \( R \) and capacitor \( C \).

\[
\frac{dU(t)}{dt} = \frac{1}{C} \left( \frac{U_0 - U}{R} - I(a, U) \right),
\]

where \( I(a, U) \) is the total current through the device. If the current density \( j \) is constant in the direction of transport (which defines the \( z \) axis), it is simply given by the integral over the cross section \( A \) of the current flow \( I = \int_A dx dy j(a(x, y), U) \). If there is charge accumulation in the sample (as in the DBRTD or the HHED, e.g.), \( I \) is given by a more complicated integral expression as shown in the Appendix. In this section we assume for simplicity that \( j \) is also homogeneous over the sample cross section, so that \( I = A j \) holds. Nevertheless, the general case can be treated analogously.

The stationary points \((a^*, U^*)\) are given by the conditions \( f(a^*, U^*) = 0 \) and \((U_0 - U^*)/R = A j(a^*, U^*)\). This depicts the operating point which is the intersection of the current-voltage characteristic \( j = j(a^*(U), U) \) with the slope of the load line \( I = U_0 - U/R \). To determine its stability we consider the temporal behavior of small fluctuations \( a(t) - a^* = \delta a e^{\lambda t} \) and \( U(t) - U^* = \delta U e^{\lambda t} \). \( \lambda \) is determined by the eigenvalue-equation

\[
\lambda \begin{pmatrix} \delta a \\ \delta U \end{pmatrix} = \mathcal{J} \begin{pmatrix} \delta a \\ \delta U \end{pmatrix},
\]

with the Jacobini

\[
\mathcal{J} = \begin{pmatrix} f_a & f_U \\ -j a A/C & -1/(RC) - j U A/C \end{pmatrix} |_{(a^*, U^*)},
\]

where the subscripts \( a \) and \( U \) denote partial derivatives. We obtain the eigenvalues

\[
\lambda_{1,2} = \frac{\text{tr} \mathcal{J}}{2} \pm \sqrt{\left( \frac{\text{tr} \mathcal{J}}{2} \right)^2 - \text{det} \mathcal{J}},
\]

where \( \text{tr} \mathcal{J} \) is the trace and \( \text{det} \mathcal{J} \) is the determinant of \( \mathcal{J} \). The operating point is stable if the real parts of both eigenvalues are negative. This is equivalent to \( \text{det} \mathcal{J} > 0 \) and \( \text{tr} \mathcal{J} < 0 \). This means:

stable operation point \( \Leftrightarrow (\text{det} \mathcal{J} > 0) \land (\text{tr} \mathcal{J} < 0) \).

A straightforward calculation of the determinant \( \text{det} \mathcal{J} \) yields

\[
\text{det} \mathcal{J} = \frac{f_a A}{C} \left( \frac{d j(a^*(U), U)}{d U} - \frac{1}{RA} \right)
\]

which is a special case of a relation derived previously.\(^{18}\)

Thus, a criterion for the stability of the operating point can be deduced from the difference between the slope of the current-voltage characteristic \( j(a^*(U), U) \) and the slope of the load line \( I = U_0 - U/R \). This has to be combined with the sign of \( f_a \) which is negative or positive on branches which are stable or unstable under voltage controlled conditions, respectively. The different possibilities for positive \( R \) and \( C \) are shown in Fig. 4. A further examination exhibits that a change of the sign of \( \text{det} \mathcal{J} \) (leading to a saddle-node bifurcation in terms of nonlinear dynamics\(^{18}\)) occurs if the slope of the load line and the characteristic are identical.

The trace is given by

\[
\text{tr} \mathcal{J} = f_a - 1/(RC) - j U A/C.
\]

If we assume, that \( j U > 0 \) holds (i.e., the negative differential conductivity is caused by the influence of the quantity \( a \)), we find for \( R, C > 0 \) the following result: \( \text{tr} \mathcal{J} \) is negative unless \( f_a > 0 \) and \( C > C_{\text{crit}} = (jaA + 1/R)/j U \). Therefore the trace becomes positive upon increasing \( C \) on the middle branch of the \( j(U) \) characteristic. For \( \text{det} \mathcal{J} > 0 \) we find a pair of complex conjugate eigenvalues \( \lambda_{1,2} \) with a positive real part above the critical value \( C_{\text{crit}} \). This depicts an oscillatory instability (an unstable focus) of the operating point via a Hopf bifurcation, which leads to self-generated oscillations if no other stable stationary point is reached in course of the temporal evolution.

IV. CURRENT FILAMENTATION

Now we want to consider the possibility that \( a(x, y, t) \) might be spatially dependent in the \( x, y \) plane perpendicular to the direction of the applied voltage. Usually there will be a term in the dynamics of \( a \) which counteracts the inhomogeneity like, e.g., a diffusional term. Then the dynamics of \( a \) is given by:

\[
\frac{\partial a(x, y, t)}{\partial t} = f(a, U) + D \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right).
\]

We restrict ourselves to a rectangular sample of size \((L_x, L_y)\) and assume Neumann boundary conditions \( \partial a/\partial x = 0 \) at \( x = 0, L_x \) and \( \partial a/\partial y = 0 \) at \( y = 0, L_y \). Then the linear stability of the homogeneous state \((a^*(x, U^*))\) is governed by the modes \( a(x, y, t) = \sum_{n_m} a_n(m) \times \cos(n \pi x L_x) \cos(m \pi y L_y) \). The amplitudes of the modes are developing according to
On branches which are stable for voltage-controlled conditions we have \( f_a < 0 \) and all spatial fluctuations are damped out. For \( f_a > 0 \) (dotted line in Fig. 1) spatial fluctuations with small wave vectors grow in time. As they do not change the total current \( I = f_a dx dy (a, U) \) they are not impeded by the circuit condition (4) in contrast to the homogeneous fluctuation. Thus, an instability occurs if one of the sample dimensions \( L_x, L_y \) is larger than \( L_{crit} = \sqrt{\pi^2 f_c} \). (Note that the precise value of \( L_{crit} \) depends on the boundary conditions but the general dependence on the length is robust.) These inhomogeneous fluctuations can lead to the formation of stable current filaments, especially if the operating point is stable against homogeneous fluctuations. Such stable current filaments constitute another branch in the current-voltage characteristic of the sample which should exhibit negative differential conductance.\(^{2,19}\) If the capacitance is increased these filaments should exhibit an oscillatory instability similar to the homogeneous mode discussed in the previous section. If the bifurcation is supercritical\(^{20}\) breathing boundaries of the filament are likely to occur. In the subcritical case spiking filaments can appear.\(^{21}\) This indicates that current or voltage oscillations may occur even if there is no oscillatory instability of the homogeneous state. More complex spatio-temporal behavior can be found if two or more competing diffusive variables are involved.\(^{3,21}\)

V. APPLICATION TO THE S- AND Z-TYPE

These general considerations can easily be applied to different devices. As an example we demonstrate the consequences for elements with an S-shaped and Z-shaped characteristic.\(^{22}\)

At first we want to consider a device exhibiting an S-shaped characteristic as sketched in Fig. 1(a). In this case the differential conductivity \( \frac{df[a(U), U]}{dU} > 0 \) on the upper and the lower branch and negative on the middle branch. Then \( Det \not= 0 \) is always positive on the upper and lower branch, while its sign depends on the slope of the load line in the middle branch. If the resistance is large enough so that the (negative) slope of the load line is larger than the (negative) differential conductivity, which corresponds effectively to current controlled conditions, the determinant is positive and the middle branch can be stable. The second condition \( tr \not= 0 \) is fulfilled unless the device is operated in the middle branch and \( C > C_{crit} \) holds. Filamentation may occur if the device is operated on the middle branch and its spatial width is larger than \( L_{crit} = \sqrt{\pi^2 f_c} \).

In conclusion the middle branch is stable for current-controlled conditions, a small total capacitance, and a small sample width. If the capacitance is increased self-generated oscillations are likely to occur, if the device does not jump to another fixed point. (The latter case seems to occur in the experiment\(^{17}\) with a quantum-dot structure.) For a large sample width and sufficiently small capacitance stable current filaments should form. They constitute a new branch in the \( I(U) \) characteristic which is usually not directly connected with the upper and lower spatially homogeneous branch.

For devices with Z-shaped characteristics as shown in Fig. 1(b) the differential conductivity \( \frac{df[a(U), U]}{dU} > 0 \) on the middle branch (where \( f_a > 0 \)) Eq. (9) indicates that the determinant is always negative there for \( R, C > 0 \). Thus, for a conventional load resistance and capacitance the middle branch of a Z-shaped characteristic cannot be stabilized. On the other hand, Eq. (9) shows that the determinant can be positive if \( C > 0 \) and \( -\frac{d j}{d U} > 1/\left(R A \right) \), or \( C < 0 \) and \( -\frac{d j}{d U} < 1/\left(R A \right) \). The condition \( tr \not= 0 \) yields:

\[
\frac{1}{RC} > - \frac{A j_a}{C} + f_a = - \frac{A}{C} \left( \frac{d j}{d U} \right) - \frac{d a^*}{d U} + f_a. \tag{13}
\]

On the middle branch we have \( f_a > 0 \) and usually \( da^*/dU < 0 \) like in Fig. 2(a). Furthermore, \( j_a < 0 \) holds as \( j(a, U) \) is monotonically decreasing in \( a \) in the Z-case as mentioned in the second section. Thus, we find

\[
- \frac{A}{C} \left( \frac{d j}{d U} \right) - \frac{d a^*}{d U} + f_a > 0, \tag{14}
\]

and condition (13) is incompatible with the condition for a positive determinant (9) in case of \( C > 0 \). Therefore we obtain the inequalities:

\[
C < 0, \tag{15}
\]

\[
- \frac{A}{R} \frac{d j}{d U} < 1 - \frac{A}{R} \frac{d j}{d U} + \frac{A j_a}{d U} + C f_a, \tag{16}
\]

as necessary and sufficient conditions for the stability of the middle branch. Considering that \( C \) is the sum of an internal sample capacitance \( C_i > 0 \) and an external capacitance \( C_{ext} \) of the circuit gives the condition

\[
C_i + C_{ext} > 1 + A f_a \frac{d a^*}{d U} - C_i. \tag{17}
\]

Indeed, such an external circuit with both \( C_{ext} < 0 \) and \( R < 0 \) has recently been realized experimentally,\(^{22}\) and it was possible to stabilize the middle branch of the Z-shaped characteristic of the DBRTD.

Regarding spatially inhomogeneous fluctuations the middle branch should become unstable for large sample cross sections. As discussed above, in this case either an additional filamentary branch or some oscillatory behavior should appear in the device.

In conclusion the stabilization of the middle branch of a Z-shaped characteristic is only possible for negative capacitance \( C_{ext} \), negative effective load resistance \( R \), and a sufficiently small sample cross sections.
VI. AN EXAMPLE OF MODELLING

In the last sections we were able to derive the conditions for stability for different cases of bistability. While we could obtain the allowed range for the slope of the load line directly from the geometrical considerations sketched in Fig. 4, we have only made qualitative statements about the restrictions of sample area and external capacitance. In order to obtain quantitative conclusions for these quantities we have to specify the functions \(j(a, U)\), \(f(a, U)\), and the parameter \(D\). In this section we want to sketch the derivation of these functions for the DBRTD as an example.

While the DBRTD usually exhibits an N-shaped characteristic for some special samples bistability associated with a Z-shaped characteristic is found. This effect is caused by the feedback of the interface charge density \(\rho_s\) (a negative quantity with units \([\text{A s/m}^2]\)) on the potential distribution and therefore the resonance condition. Thus we can easily identify the essential physical quantity to be the interface charge density and set \(a = \rho_s\). Now we can describe the transport in the DBRTD within the sequential tunnelling approach and calculate the current-density \(j_1\) from the emitter into the well and the current-density \(j_2\) from the well into the collector as functions of the total voltage \(U\) and the charge density \(\rho_s\). This has been done in Ref. 15, e.g. From the continuity equation we obtain:

\[
\frac{\partial \rho_s(x,y,t)}{\partial t} = j_1 - j_2 - \nabla \cdot J,
\]

where \(J\) is the surface current within the \((x,y)\) plane of the quantum well. It can be modelled by a drift-diffusion approach:

\[
J = \mu \rho_s F - D_{\text{diff}} \nabla \rho_s,
\]

where \(\mu\) is the mobility and \(D_{\text{diff}}\) the diffusion constant of the electrons in the quantum well. The electric field \(F\) along the well plane is determined by

\[
F = \frac{-1}{\epsilon/L_1 + \epsilon/L_2} \nabla \rho_s
\]

in the quasistationary state if \(\rho_s\) only varies on a length scale much larger than \(L_1 + L_2\). Here \(\epsilon\) is the dielectric constant of the material and \(L_1\) and \(L_2\) are the widths of the insulating regions on both sides of the quantum well. Thus, \(L_1\) is roughly given by the width of the first barrier and \(L_2\) by the width of the second barrier plus the depleted region near the absorbing contact. For small deviations from the homogeneous state \(\rho_s^0 = \text{const}\) we therefore recover Eq. (11) with

\[
f(\rho_s, U) = j_1 - j_2,
\]

\[
D = \frac{\mu |\rho_s^0|}{\epsilon L_1 + \epsilon L_2} + D_{\text{diff}}.
\]

The current is obtained from Eq. (A5) of the Appendix by setting \(j_z = j_1\) for \(0 < z < L_1\) and \(j_z = j_2\) for \(L_1 < z < L_1 + I_2 = I\).

\[
I(t) = \int_A dy \frac{L_1 j_1 + L_2 j_2}{L_1 + L_2}.
\]

Therefore all quantities are defined enabling the calculation of the ranges of stability and the dynamical behaviour. Note that this approach is completely analogous to the description of the HHED in Ref. 16. The difference in that the HHED exhibits an S-shaped and the DBRTD Z-shaped characteristics is only reflected in the different forms of the functions \(j_1(\rho_s, U)\) and \(j_2(\rho_s, U)\).

VII. CONCLUDING REMARKS

The stability of S-shaped elements has often been discussed regarding circuits with inductive and capacitive elements (see, e.g., Ref. 23). These approaches assume that there is a specific relation \(j(U)\) between the current and the voltage in the element, which can be multivalued, but nevertheless does exist. Thus, they neglect any internal degree of freedom. The oscillatory behavior is caused by the \(L - C\) resonator which is amplified by the negative differential conductivity of the device. Similarly, current filamentation is treated by assuming that the voltage along the sample varies by inductive effects within the device (see chapter 7.7 in Ref. 1). The use of the relation \(j(U)\) implies that the internal degree of freedom is always at its stationary value \(a^*(U)\). This has two severe implications. First, the operating point is required to be stable with respect to the internal dynamics. Second, the internal dynamics must relax to this stable point on a time scale which is much faster than the relevant time scales considered in the problem. This demonstrates that such an approach using \(j(U)\) can only be valid if the stability with respect to the internal dynamics has already been proved. As shown above the properties associated with this internal degree of freedom exhibit the well-known typical features like self-sustained oscillations and current filamentation. This indicates that the combination of an external circuit conditions with the internal dynamics is the essential reason for the instabilities found experimentally.

In case of the DBRTD the dynamics is often discussed in terms of quantum inductances whose origin seems not to be clearly understood. This approach has been used in Ref. 22 to derive the conditions for stabilizing the middle branch of the DBRTD which are very similar to our conditions. Notebom et al. have shown recently that the results of the phenomenological quantum conductance approach correspond to a microscopic dynamics which has just the structure presented in this paper. This indicates that the quantum inductance is nothing but a kind of equivalent circuit for the internal dynamics in a certain range of operation. Indeed, every inductance \(L\) adds an additional degree of freedom through the differential equation \(dI/dt = UI/L\) to the circuit.

In this paper we have considered the simplest case of a single relevant dynamic variable and a circuit containing only capacitive elements. If there is more than one relevant internal degree of freedom the situation becomes more complicated and chaotic behavior may occur in the homogeneous system, too (see Ref. 18). Another important feature is the inclusion of noise. This can lead to a suppression of the
metastable states near the bifurcations so that even the full lines of the characteristics shown in Fig. 1 will not be observed in full length.

In conclusion we have shown that a simple approach in the framework of a single internal degree of freedom can reproduce many features seen experimentally. This shows that a theoretical modelling of such devices must include a dynamic treatment of at least one internal variable in order to obtain good agreement with experiments.

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APPENDIX

In this appendix we determine the dynamics of the voltage in the case of an inhomogeneous current distribution along the sample leading to charge accumulation effects. The voltage is applied in the z-direction. We restrict ourselves to the important case of a layered structure, so that the material properties like the dielectric constant $\varepsilon(z)$ are only $z$-dependent. Ampère’s law gives $\text{div}(\mathbf{e} \mathbf{E} / dt + \mathbf{j}) = 0$, where $\mathbf{E}$ is the electric field. From this we obtain:

$$\int_{\partial V} d\mathbf{A} \cdot \left( e \frac{d\mathbf{F}}{dt} + \mathbf{j} \right) = 0. \quad (A1)$$

Now we integrate over the surface $\partial V$ shown in Fig. 5 and find

$$C_{\text{ext}} \frac{dU}{dt} + \int_{A} dx \ dy \left( \frac{dF_{z}}{dt} + \frac{j_{z}}{\varepsilon(z)} \right) - \frac{I_{0}}{\varepsilon(z)} = 0, \quad (A2)$$

where $C_{\text{ext}}$ is the external capacitance, $I_{0}$ is the total current and $z$ is the position of the intersection of the surface with the sample of cross-section $A$. Here we have neglected the electric fields outside the sample which is reasonable for structures whose diameter is much larger than the sample length $L$ in $z$-direction. Integrating over the $z$-direction we obtain

$$\left( A + C_{\text{ext}} \int_{0}^{L} dz \frac{1}{\varepsilon(z)} \right) \frac{dU}{dt} + \int_{0}^{L} dz \int_{A} dx \ dy \frac{j_{z}}{\varepsilon(z)} - \int_{0}^{L} dz \frac{I_{0}}{\varepsilon(z)} = 0 \quad (A3)$$

as $\int_{0}^{L} dz F_{z} = U$ is not dependent on $(x,y)$, if we assume ideal metallic planar contacts at $z = 0$ and $z = L$ and neglect time varying magnetic fields. Defining the intrinsic sample capacitance $C_{s}$ by $C_{s}^{-1} = \int_{0}^{L} dz [A e(z)]^{-1}$ we obtain:

$$\frac{dU}{dt} = \frac{1}{C_{s} + C_{\text{ext}}} \left( \frac{C_{s}}{I_{0}} \int_{A} dx \ dy \int_{0}^{L} dz \frac{j_{z}}{\varepsilon(z)} \right). \quad (A4)$$

From this expression we can easily identify the quantity

$$I = \frac{C_{s}}{A} \int_{0}^{L} dz \frac{j_{z}}{\varepsilon(z)} \quad (A5)$$

to be included in Eq. (4).