Examination I for PHYS 6220/7220, Fall 2013

1. Three points P_1, P_2 and P_3 are fixed and equally spaced about the circumference of a circle. Each point exerts a force on a mass m given by $\mathbf{F}_i = -k\mathbf{r}_i$ where $\mathbf{r}_i$ is the position vector of the mass with respect to the point P_i, with i = 1, 2 and 3. Here k is positive constant of suitable dimensions. At t = 0 the position and velocity of the mass are **S** and **W**, respectively. There is no gravity in this problem.

(a) Define clearly a frame of reference and a set of generalized coordinates for the system with a figure. State how the coordinate system and all coordinates are defined in words as well. (**2 points**)

(b) Derive the Lagrangian for the system. (**2 points**)

(c) Derive all the Euler-Lagrange equations of motion. (**2 points**)

(d) Solve all the Euler-Lagrange equations including all given initial conditions. (**2 points**)

(e) State all constants of motion in the problem. Explain why these constants are as expected. (**2 points**)

(f) Under what conditions can the particle motion be limited to occur on a sphere? (**1 point**)

2. A river has fixed width w with one side along the Y axis. A boat starts from the origin O on one side of the river and travels to the other side at a point P with coordinates (w, b), where b > 0. The boat travels at a constant natural speed, v_0, if the water is still. The river current has a velocity that is a vector function, $\mathbf{V_C} = [v_0 g(x)]\ \mathbf{j}$, which does not change with time. Here **j** is the unit vector along the Y axis. The function g(x) is smoothly varying and such that g(0) = g(w) = 0 and $0 \leq g(x) < 1, \forall x \in [0, w]$.

(a) Draw a figure showing the trajectory of the boat. Let the shape of this trajectory be given by the unknown function f(x). Define the angle, β, that the tangent to the trajectory makes with the X axis at any arbitrary location. Note $\beta = \beta(x)$. Define the steering angle that the rudder of the boat makes with the X axis as α. Note $\alpha = \alpha(x)$. (**1 point**)

(b) Write an expression for the time of travel, T, of the boat using only the relevant quantities amongst w, b, v_0, g(x), f(x), and the two angles α and β. (**2 points**)

(c) Write an expression that can relate the two angles with other relevant quantities including f(x). (**2 points**)

(d) From results in parts (b) and (c) get rid of α and β from the time of travel expression to only have it dependent on f(x) and g(x) and their derivatives if needed. (**2 points**)

(e) Applying the calculus of variations find the equation that f(x) should obey to minimize the time of travel. (**2 points**)

(f) Solve this equation as much as you can. (**1 point**)

(g) Use all the boundary conditions given. What is the form of f(x) in the limiting case g(x) = k, a constant? Find the time of travel for the special case k = 0. Justify your answer in this latter case. (**2 points**)