being time dependent, with the rotation axis along z and the wire in the xy plane. The transformation equations explicitly contain the time.

$$x = r\cos\omega t, \qquad (\omega = \text{angular velocity of rotation})$$

$$y = r\sin\omega t. \qquad (r = \text{distance along wire from rotation axis})$$

While we could then find T (here the same as L) by the same procedure used to obtain (1.71), it is simpler to take over (1.75) directly, expressing the constraint by the relation $\dot{\theta} = \omega$:

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right).$$

Note that T is not a homogeneous quadratic function of the generalized velocities, since there is now a term not involving $\dot{r}$. The equation of motion is then

$$m\ddot{r} - mr\omega^2 = 0$$

or

$$\ddot{r} = r\omega^2,$$

which is the familiar simple harmonic oscillator equation with a change of sign. The solution $r = e^{\omega t}$ for a bead initially at rest on the wire shows that the bead moves exponentially outwards. Again, the method cannot furnish the force of constraint that keeps the bead on the wire. Equation (1.26) with the angular momentum, $L = mr^2\omega = m\omega r_0^2 e^{2\omega t}$, provides the force $F = N/r$, which produces the constraint force, $F = 2mr_0\omega^2 e^{\omega t}$, acting perpendicular to the wire and the axis of rotation.

DERIVATIONS

1. Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F}\cdot\mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2.$$

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

9. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(\mathbf{r}, \mathbf{t}),$$

$$\phi \to \phi - \frac{1}{c}\frac{\partial\psi}{\partial t},$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

10. Let $q_1, \ldots, q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, \ldots, s_n$ by means of transformation equations

$$q_i = q_i(s_1, \ldots, s_n, t), \qquad i = 1, \ldots, n.$$

(Such a transformation is called a *point transformation.*) Show that if the Lagrangian function is expressed as a function of s_j, $\dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

EXERCISES

11. Consider a uniform thin disk that rolls without slipping on a horizontal plane. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk.

    (a) Derive Lagrange's equations and find the generalized force.

    (b) Discuss the motion if the force is not applied parallel to the plane of the disk.

12. The *escape velocity* of a particle on Earth is the minimum velocity required at Earth's surface in order that the particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for Earth, ignoring the presence of the Moon, is 11.2 km/s.

13. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational

field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

15. A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma} \cdot \mathbf{L},$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and $\boldsymbol{\sigma}$ is a fixed vector in space.

(a) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq. (1.58).

(b) Show that the components in the two coordinate systems are related to each other as in Eq. (1.49).

(c) Obtain the equations of motion in spherical polar coordinates.

16. A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right),$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics.)

17. A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg·m/s.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg what is its kinetic energy, in electron volts?

18. A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{K}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$.