Physics 4210/5210 Theoretical Mechanics Fall 2023 – Exam 3

Total points for the exam: 16

1. A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in Figure 1.
(**a**) Assuming that the angle ϕ remains small, write down the system's Lagrangian and the equations of motion for x and ϕ. [**2 points**]
(**b**) Assuming that m = M = L= g = 1 and k = 2 (all in appropriate units) find the normal frequencies, and for each normal frequency find and describe the motion of the corresponding normal mode. [**2 points**]

2. (**a**) Write down the integral for the moment of inertia of a uniform cube of side and mass M, rotating about an edge, and show that it is equal to $(2Ma^2)/3$. [**2 points**]
(b) If I balance the cube on an edge in unstable equilibrium on a rough table, it will eventually topple and rotate until it hits the table. By considering the energy of the cube, find its angular velocity just before it hits the table. Assume the edge does not slide onto the table. [**3 points**]

3. On a certain planet, which is perfectly spherically symmetric, the free-fall acceleration has magnitude $g = g_o$ at the North Pole and $g = \lambda g_o$ at the equator (with $0 \leq \lambda \leq 1$). Find $g(\theta)$, the freefall acceleration at colatitude θ as a function of θ. [**3 points**]

4. (**a**) By examining the effective potential energy, find the radius at which a planet (or comet) with angular momentum ℓ can orbit the sun in a circular orbit with fixed radius. [**2 points**]
(b) Show that this circular orbit is stable, in the sense that a small radial nudge will cause only small radial oscillations. Show that the period of these oscillations is equal to the planet's orbital period. [**2 points**]

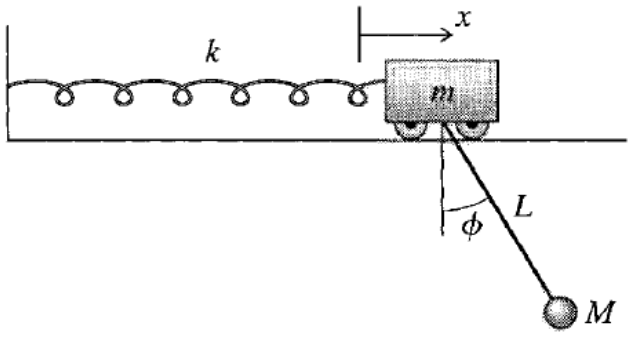

Figure 1.