

This is a printout of the wiki article on rotating frames. Read carefully. Notice that the Euler force is not addressed in our text.

All [non-inertial reference frames](#) exhibit [fictitious forces](#). Rotating reference frames are characterized by three fictitious forces^[1]

- the [centrifugal force](#)
- the [Coriolis force](#)

and, for non-uniformly rotating reference frames,

- the [Euler force](#).

Scientists living in a rotating box can measure the speed and direction of their rotation by measuring these [fictitious forces](#). For example, [Léon Foucault](#) was able to show the [Coriolis force](#) that results from the Earth's rotation using the [Foucault pendulum](#). If the Earth were to rotate many times faster, these fictitious forces could be felt by humans, as they are when on a spinning [carousel](#).

Relating rotating frames to stationary frames

The following is a derivation of the formulas for accelerations as well as fictitious forces in a rotating frame. It begins with the relation between a particle's coordinates in a rotating frame and its coordinates in an inertial (stationary) frame. Then, by taking time derivatives, formulas are derived that relate the velocity of the particle as seen in the two frames, and the acceleration relative to each frame. Using these accelerations, the fictitious forces are identified by comparing Newton's second law as formulated in the two different frames.

Relation between positions in the two frames

To derive these fictitious forces, it's helpful to be able to convert between the coordinates (x', y', z') of the rotating reference frame and the coordinates (x, y, z) of an [inertial reference frame](#) with the same origin. If the rotation is about the z axis with an [angular velocity](#) Ω and the two reference frames coincide at time $t = 0$, the transformation from rotating coordinates to inertial coordinates can be written

$$\begin{aligned}x &= x' \cos(\Omega t) - y' \sin(\Omega t) \\y &= x' \sin(\Omega t) + y' \cos(\Omega t)\end{aligned}$$

whereas the reverse transformation is

$$x' = x \cos(-\Omega t) - y \sin(-\Omega t)$$

$$y' = x \sin(-\Omega t) + y \cos(-\Omega t)$$

This result can be obtained from a [rotation matrix](#).

Introduce the unit vectors \hat{i} , \hat{j} , \hat{k} representing standard unit basis vectors in the rotating frame. The time-derivatives of these unit vectors are found next. Suppose the frames are aligned at $t = 0$ and the z -axis is the axis of rotation. Then for a counterclockwise rotation through angle Ωt :

$$\hat{i}(t) = (\cos \Omega t, \sin \Omega t)$$

where the (x, y) components are expressed in the stationary frame. Likewise,

$$\hat{j}(t) = (-\sin \Omega t, \cos \Omega t) .$$

Thus the time derivative of these vectors, which rotate without changing magnitude, is

$$\begin{aligned} \frac{d}{dt} \hat{i}(t) &= \Omega(-\sin \Omega t, \cos \Omega t) = \Omega \hat{j} ; \\ \frac{d}{dt} \hat{j}(t) &= \Omega(-\cos \Omega t, -\sin \Omega t) = -\Omega \hat{i} . \end{aligned}$$

This result is the same as found using a [vector cross product](#) with the rotation vector $\boldsymbol{\Omega}$ pointed along the z -axis of rotation $\boldsymbol{\Omega} = (0, 0, \Omega)$, namely,

$$\frac{d}{dt} \hat{u} = \boldsymbol{\Omega} \times \hat{u} ,$$

where \hat{u} is either \hat{i} or \hat{j} .

Time derivatives in the two frames

Introduce the unit vectors \hat{i} , \hat{j} , \hat{k} representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at the speed of $\boldsymbol{\Omega}$ about an axis $\boldsymbol{\Omega}$ then each unit vector \hat{u} of the rotating coordinate system abides by the following equation:

$$\frac{d}{dt} \hat{u} = \boldsymbol{\Omega} \times \hat{u} .$$

Then if we have a vector function \mathbf{f} ,

$$\mathbf{f}(t) = f_x(t)\hat{i} + f_y(t)\hat{j} + f_z(t)\hat{k} ,$$

and we want to examine its first derivative we have (using the [product rule](#) of differentiation):^{[2][3]}

$$\begin{aligned} \frac{d}{dt} \mathbf{f} &= \frac{df_x}{dt} \hat{\mathbf{i}} + \frac{d\hat{\mathbf{i}}}{dt} f_x + \frac{df_y}{dt} \hat{\mathbf{j}} + \frac{d\hat{\mathbf{j}}}{dt} f_y + \frac{df_z}{dt} \hat{\mathbf{k}} + \frac{d\hat{\mathbf{k}}}{dt} f_z \\ &= \frac{df_x}{dt} \hat{\mathbf{i}} + \frac{df_y}{dt} \hat{\mathbf{j}} + \frac{df_z}{dt} \hat{\mathbf{k}} + [\boldsymbol{\Omega} \times (f_x \hat{\mathbf{i}} + f_y \hat{\mathbf{j}} + f_z \hat{\mathbf{k}})] \\ &= \left(\frac{d\mathbf{f}}{dt} \right)_r + \boldsymbol{\Omega} \times \mathbf{f}(t) , \end{aligned}$$

where $\left(\frac{d\mathbf{f}}{dt} \right)_r$ is the rate of change of \mathbf{f} as observed in the rotating coordinate system. As a shorthand the differentiation is expressed as:

$$\frac{d}{dt} \mathbf{f} = \left[\left(\frac{d}{dt} \right)_r + \boldsymbol{\Omega} \times \right] \mathbf{f} .$$

This result is also known as the Transport Theorem in analytical dynamics, and is also sometimes referred to as the Basic Kinematic Equation.^[4]

Relation between velocities in the two frames

A velocity of an object is the time-derivative of the object's position, or

$$\mathbf{v} \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{dt}$$

The time derivative of a position $\mathbf{r}(t)$ in a rotating reference frame has two components, one from the explicit time dependence due to motion of the particle itself, and another from the frame's own rotation. Applying the result of the previous subsection to the displacement $\mathbf{r}(t)$, the [velocities](#) in the two reference frames are related by the equation

$$\mathbf{v}_i \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{dt} = \left(\frac{d\mathbf{r}}{dt} \right)_r + \boldsymbol{\Omega} \times \mathbf{r} = \mathbf{v}_r + \boldsymbol{\Omega} \times \mathbf{r} ,$$

where subscript i means the inertial frame of reference, and r means the rotating frame of reference.

Relation between accelerations in the two frames

Acceleration is the second time derivative of position, or the first time derivative of velocity

$$\mathbf{a}_i \stackrel{\text{def}}{=} \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_i = \left(\frac{d\mathbf{v}}{dt} \right)_i = \left[\left(\frac{d}{dt} \right)_r + \boldsymbol{\Omega} \times \right] \left[\left(\frac{d\mathbf{r}}{dt} \right)_r + \boldsymbol{\Omega} \times \mathbf{r} \right],$$

where subscript i means the inertial frame of reference. Carrying out the [differentiations](#) and re-arranging some terms yields the acceleration in the *rotating* reference frame

$$\mathbf{a}_r = \mathbf{a}_i - 2\boldsymbol{\Omega} \times \mathbf{v}_r - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}$$

where $\mathbf{a}_r \stackrel{\text{def}}{=} \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_r$ is the apparent acceleration in the rotating reference frame, the term $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ represents [centrifugal acceleration](#), and the term $-2\boldsymbol{\Omega} \times \mathbf{v}_r$ is the [Coriolis effect](#).

Newton's second law in the two frames

When the expression for acceleration is multiplied by the mass of the particle, the three extra terms on the right-hand side result in [fictitious forces](#) in the rotating reference frame, that is, apparent forces that result from being in a [non-inertial reference frame](#), rather than from any physical interaction between bodies.

Using [Newton's second law of motion](#) $\mathbf{F} = m\mathbf{a}$, we obtain: [\[1\]\[2\]\[3\]\[5\]\[6\]](#)

- the [Coriolis force](#)

$$\mathbf{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \mathbf{v}_r$$

- the [centrifugal force](#)

$$\mathbf{F}_{\text{centrifugal}} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

- and the [Euler force](#)

$$\mathbf{F}_{\text{Euler}} = -m \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}$$

where m is the mass of the object being acted upon by these [fictitious forces](#). Notice that all three forces vanish when the frame is not rotating, that is, when $\boldsymbol{\Omega} = \mathbf{0}$.

For completeness, the inertial acceleration \mathbf{a}_i due to impressed external forces \mathbf{F}_{imp} can be determined from the total physical force in the inertial (non-rotating) frame (for example, force from physical interactions such as [electromagnetic forces](#)) using [Newton's second law](#) in the inertial frame:

$$\mathbf{F}_{\text{imp}} = m\mathbf{a}_i$$

Newton's law in the rotating frame then becomes

$$\mathbf{F}_r = \mathbf{F}_{\text{imp}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{Euler}} = m\mathbf{a}_r .$$

In other words, to handle the laws of motion in a rotating reference frame: