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## Spherical coordinates #rvs

The spherical coordinate system extends polar coordinates into 3D by using an angle  $\phi$  for the third coordinate. This gives coordinates  $(r, \theta, \phi)$  consisting of:

coordinate	name	range	definition
$r$	radius	$0 \leq r < \infty$	distance from the origin
$\theta$	azimuth	$-\pi < \theta \leq \pi$	angle from the $x$ -axis in the $x$ - $y$ plane
$\phi$	inclination	$0 \leq \phi \leq \pi$	angle from the positive $z$ axis

The diagram below shows the spherical coordinates of a point  $P$ . By changing the display options, we can see that the basis vectors are tangent to the corresponding coordinate lines. Changing  $\theta$  moves  $P$  along the  $\theta$  coordinate line in the direction  $\hat{e}_\theta$ , and similarly for the other coordinates.

Spherical coordinates are defined with respect to a set of Cartesian coordinates, and can be converted to and from these coordinates using the [atan2](#) function as follows.

### Conversion between spherical and Cartesian coordinates #rvs-ec

$$\begin{aligned} x &= r \cos \theta \sin \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \text{atan2}(y, x) \\ z &= r \cos \phi & \phi &= \arccos(z/r) \end{aligned}$$

Derivation #rvs-ec-d +

The basis vectors are tangent to the coordinate lines and form an orthonormal basis  $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$  that depends on the current position  $P$  as follows.

### Spherical basis vectors #rvs-eb

$$\begin{aligned} \hat{e}_r &= \cos \theta \sin \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \phi \hat{k} \\ \hat{e}_\theta &= -\sin \theta \hat{i} + \cos \theta \hat{j} \\ \hat{e}_\phi &= \cos \theta \cos \phi \hat{i} + \sin \theta \cos \phi \hat{j} - \sin \phi \hat{k} \\ \hat{i} &= \cos \theta \sin \phi \hat{e}_r - \sin \theta \hat{e}_\theta + \cos \theta \cos \phi \hat{e}_\phi \\ \hat{j} &= \sin \theta \sin \phi \hat{e}_r + \cos \theta \hat{e}_\theta + \sin \theta \cos \phi \hat{e}_\phi \\ \hat{k} &= \cos \phi \hat{e}_r - \sin \phi \hat{e}_\phi \end{aligned}$$

Derivation #rvs-eb-d +

+ **Warning:**  $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$  is not right-handed #rvs-wt

If the spherical coordinates change with time then this causes the spherical basis vectors to rotate with the following angular velocity.

### Angular velocity of the spherical basis #rvs-ew

$$\begin{aligned} \vec{\omega} &= \dot{\phi} \hat{e}_\theta + \dot{\theta} \hat{k} \\ &= \dot{\theta} \cos \phi \hat{e}_r + \dot{\phi} \hat{e}_\theta - \dot{\theta} \sin \phi \hat{e}_\phi \end{aligned}$$

Derivation #rvs-ew-d +

The rotation of the basis vectors caused by changing coordinates can be directly computed, giving the time derivatives below.

### Notation note #rvs-ic

There are many different conventions for spherical coordinates notation, so it's important to check which variant is being used in any document. The convention used here is common in mathematics. In physics it is also common to use the same angles, but to reverse the symbol convention so that  $\phi$  is the azimuth and  $\theta$  is the inclination. Many different names for the coordinates are also used, with the inclination also being called the *zenith angle*, *polar angle*, or *normal angle*. Alternatively, the inclination may be replaced by the *elevation* measured up from the  $x$ - $y$  plane. In geographic coordinates the azimuth is the *longitude*  $\lambda$  and the elevation is the *latitude*  $\phi$ , while in celestial coordinates the azimuth is the *right ascension*  $\alpha$  and the elevation is the *declination*  $\delta$ .

The angles themselves may be given in radians or degrees, and are sometimes written in degrees (symbol  $^\circ$ ), arc-minutes (1/60 of a degree, symbol  $'$ ) and arc-seconds (1/60 of an arc-minute, symbol  $''$ ). Positive and negative angles are normally indicated by  $+/-$ , but sometimes use North/South (N/S) or East/West (E/W). Thus a longitude may be written as  $\lambda = 88^\circ 12' 15'' \text{ W} = 88.2042^\circ \text{ W} = -88.2042^\circ = -1.53945 \text{ rad}$ .

### Related applications #rvs-ia

- [Shortest flight paths](#)

**Time derivatives of spherical basis vectors** #rvs-et

$$\begin{aligned}\dot{\hat{e}}_r &= \dot{\theta} \sin \phi \hat{e}_\theta + \dot{\phi} \hat{e}_\phi \\ \dot{\hat{e}}_\theta &= -\dot{\theta} \sin \phi \hat{e}_r - \dot{\theta} \cos \phi \hat{e}_\phi \\ \dot{\hat{e}}_\phi &= -\dot{\phi} \hat{e}_r + \dot{\theta} \cos \phi \hat{e}_\theta\end{aligned}$$

**Derivation** #rvs-et-d +

A point  $P$  at a time-varying position  $(r, \theta, \phi)$  has position vector  $\vec{r}$ , velocity  $\vec{v} = \dot{\vec{r}}$ , and acceleration  $\vec{a} = \ddot{\vec{r}}$  given by the following expressions in spherical components.

**Position, velocity, and acceleration in spherical components** #rvs-ep

$$\begin{aligned}\vec{r} &= r \hat{e}_r \\ \vec{v} &= \dot{r} \hat{e}_r + r\dot{\theta} \sin \phi \hat{e}_\theta + r\dot{\phi} \hat{e}_\phi \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2 \sin^2 \phi - r\dot{\phi}^2) \hat{e}_r \\ &\quad + (r\ddot{\theta} \sin \phi + 2\dot{r}\dot{\theta} \sin \phi + 2r\dot{\theta}\dot{\phi} \cos \phi) \hat{e}_\theta \\ &\quad + (r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin \phi \cos \phi) \hat{e}_\phi\end{aligned}$$

**Derivation** #rvs-ep-d +