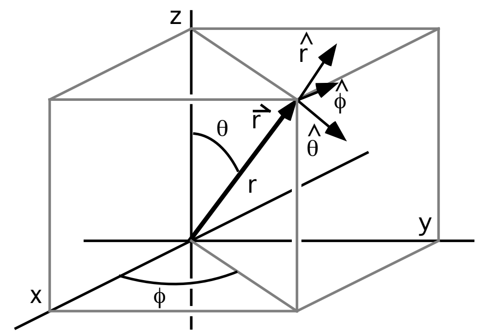
# Spherical Coordinates




## Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right)$$
$$\phi = \arctan(y, x)$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

## Unit Vectors

The unit vectors in the spherical coordinate system are functions of position. It is convenient to express them in terms of the *spherical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$
$$\hat{\phi} = \frac{\hat{z} \times \hat{r}}{\sin\theta} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{\theta} = \hat{\phi} \times \hat{r} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

## Variations of unit vectors with the coordinates

Using the expressions obtained above it is easy to derive the following handy relationships:

$$\frac{\partial\hat{r}}{\partial r} = 0$$
$$\frac{\partial\hat{r}}{\partial\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta = \hat{\theta}$$
$$\frac{\partial\hat{r}}{\partial\phi} = -\hat{x}\sin\theta\sin\phi + \hat{y}\sin\theta\cos\phi = \left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right)\sin\theta = \hat{\phi}\sin\theta$$

$$\frac{\partial\hat{\phi}}{\partial r} = 0$$
$$\frac{\partial\hat{\phi}}{\partial\theta} = 0$$
$$\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{x}\cos\phi - \hat{y}\sin\phi = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)$$

$$\frac{\partial\hat{\theta}}{\partial r} = 0$$
$$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{x}\sin\theta\cos\phi - \hat{y}\sin\theta\sin\phi - \hat{z}\cos\theta = -\hat{r}$$
$$\frac{\partial\hat{\theta}}{\partial\phi} = -\hat{x}\cos\theta\sin\phi + \hat{y}\cos\theta\cos\phi = \hat{\phi}\cos\theta$$

## Path increment

We will have many uses for the path increment $d\vec{r}$ expressed in spherical coordinates:

$$d\vec{r} = d\left(r\hat{r}\right) = \hat{r}dr + rd\hat{r} = \hat{r}dr + r\left(\frac{\partial \hat{r}}{\partial r}dr + \frac{\partial \hat{r}}{\partial \theta}d\theta + \frac{\partial \hat{r}}{\partial \phi}d\phi\right)$$

$$= \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

## Time derivatives of the unit vectors

We will also have many uses for the time derivatives of the unit vectors expressed in spherical coordinates:

$$\dot{\hat{r}} = \frac{\partial \hat{r}}{\partial r}\dot{r} + \frac{\partial \hat{r}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{r}}{\partial \phi}\dot{\phi} = \hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta$$

$$\dot{\hat{\theta}} = \frac{\partial \hat{\theta}}{\partial r}\dot{r} + \frac{\partial \hat{\theta}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\theta}}{\partial \phi}\dot{\phi} = -\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta$$

$$\dot{\hat{\phi}} = \frac{\partial \hat{\phi}}{\partial r}\dot{r} + \frac{\partial \hat{\phi}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\phi}}{\partial \phi}\dot{\phi} = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\phi}$$

## Velocity and Acceleration

The velocity and acceleration of a particle may be expressed in spherical coordinates by taking into account the associated rates of change in the unit vectors:

$$\vec{v} = \dot{\vec{r}} = \dot{\hat{r}}r + \hat{r}\dot{r}$$

$$\boxed{\vec{v} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta}$$

$$\vec{a} = \dot{\vec{v}} = \dot{\hat{r}}\dot{r} + \hat{r}\ddot{r} + \dot{\hat{\theta}}r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta} + \dot{\hat{\phi}}r\dot{\phi}\sin\theta + \hat{\phi}\dot{r}\dot{\phi}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta$$

$$= \left(\hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta\right)\dot{r} + \hat{r}\ddot{r} + \left(-\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta\right)r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta}$$

$$+\left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\phi}\right]r\dot{\phi}\sin\theta + \hat{\phi}\dot{r}\dot{\phi}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta$$

$$\boxed{\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin\theta\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right) + \hat{\phi}\left(r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta\right)}$$

## The del operator from the definition of the gradient

Any (static) scalar field $u$ may be considered to be a function of the spherical coordinates $r$, $\theta$, and $\phi$. The value of $u$ changes by an infinitesimal amount $du$ when the point of observation is changed by $d\vec{r}$. That change may be determined from the partial derivatives as

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi\ .$$

But we also define the gradient in such a way as to obtain the result

$$du = \vec{\nabla}u \cdot d\vec{r}$$

Therefore,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi = \vec{\nabla}u \cdot d\vec{r}$$

or, in spherical coordinates,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi = \left(\vec{\nabla}u\right)_r dr + \left(\vec{\nabla}u\right)_\theta r d\theta + \left(\vec{\nabla}u\right)_\phi r\sin\theta\, d\phi$$

and we demand that this hold for any choice of $dr$, $d\theta$, and $d\phi$. Thus,

$$\left(\vec{\nabla}u\right)_r = \frac{\partial u}{\partial r}, \quad \left(\vec{\nabla}u\right)_\theta = \frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \left(\vec{\nabla}u\right)_\phi = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \theta},$$

from which we find

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}}$$

## Divergence

The divergence $\vec{\nabla}\cdot\vec{A}$ is carried out taking into account, once again, that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}\right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\vec{A} \\
&= \hat{r}\cdot\frac{\partial\vec{A}}{\partial r} + \frac{\hat{\theta}}{r}\cdot\frac{\partial\vec{A}}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial\vec{A}}{\partial \phi} \\
&= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + A_r\frac{\partial\hat{r}}{\partial r} + A_\theta\frac{\partial\hat{\theta}}{\partial r} + A_\phi\frac{\partial\hat{\phi}}{\partial r}\right) \\
&\quad + \frac{\hat{\theta}}{r}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\frac{\partial\hat{r}}{\partial \theta} + A_\theta\frac{\partial\hat{\theta}}{\partial \theta} + A_\phi\frac{\partial\hat{\phi}}{\partial \theta}\right) \\
&\quad + \frac{\hat{\phi}}{r\sin\theta}\cdot\left(\frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\frac{\partial\hat{r}}{\partial \phi} + A_\theta\frac{\partial\hat{\theta}}{\partial \phi} + A_\phi\frac{\partial\hat{\phi}}{\partial \phi}\right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + 0 + 0 + 0\right) \\
&\quad + \frac{\hat{\theta}}{r}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\hat{\theta} + A_\theta\left(-\hat{r}\right) + 0\right) \\
&\quad + \frac{\hat{\phi}}{r\sin\theta}\cdot\left(\frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\sin\theta\hat{\phi} + A_\theta\cos\theta\hat{\phi} + A_\phi\left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\right]\right) \\
&= \left(\frac{\partial A_r}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r}\right) + \left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} + \frac{A_r}{r} + \frac{A_\theta\cos\theta}{r\sin\theta}\right) \\
&= \left(\frac{\partial A_r}{\partial r} + \frac{2A_r}{r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_\theta\cos\theta}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}
\end{aligned}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

## Curl

The curl $\vec{\nabla} \times \vec{A}$ is also carried out taking into account that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec{\nabla}\times \vec{A} = \left( \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \times \left( A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi} \right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}
\vec{\nabla}\times \vec{A} &= \left( \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \times \vec{A} \\
&= \hat{r}\times\frac{\partial \vec{A}}{\partial r} + \frac{\hat{\theta}}{r}\times\frac{\partial \vec{A}}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\times\frac{\partial \vec{A}}{\partial \phi} \\
&= \hat{r}\times\left( \frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial r} + A_\theta\frac{\partial \hat{\theta}}{\partial r} + A_\phi\frac{\partial \hat{\phi}}{\partial r} \right) \\
&\quad + \frac{\hat{\theta}}{r}\times\left( \frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial \theta} + A_\theta\frac{\partial \hat{\theta}}{\partial \theta} + A_\phi\frac{\partial \hat{\phi}}{\partial \theta} \right) \\
&\quad + \frac{\hat{\phi}}{r\sin\theta}\times\left( \frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial \phi} + A_\theta\frac{\partial \hat{\theta}}{\partial \phi} + A_\phi\frac{\partial \hat{\phi}}{\partial \phi} \right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\vec{\nabla}\times \vec{A} &= \hat{r}\times\left( \frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + 0 + 0 + 0 \right) \\
&\quad + \frac{\hat{\theta}}{r}\times\left( \frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\hat{\theta} + A_\theta(-\hat{r}) + 0 \right) \\
&\quad + \frac{\hat{\phi}}{r\sin\theta}\times\left( \frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\sin\theta\hat{\phi} + A_\theta\cos\theta\hat{\phi} + A_\phi\left[ -\left( \hat{r}\sin\theta + \hat{\theta}\cos\theta \right) \right] \right) \\
&= \left( \frac{\partial A_\theta}{\partial r}\hat{\phi} - \frac{\partial A_\phi}{\partial r}\hat{\theta} \right) + \left( -\frac{1}{r}\frac{\partial A_r}{\partial \theta}\hat{\phi} + \frac{1}{r}\frac{\partial A_\phi}{\partial \theta}\hat{r} + \frac{A_\theta}{r}\hat{\phi} \right) \\
&\quad + \left( \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\hat{r} - \frac{A_\phi}{r}\hat{\theta} + \frac{A_\phi\cos\theta}{r\sin\theta}\hat{r} \right) \\
&= \hat{r}\left( \frac{1}{r}\frac{\partial A_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi} + \frac{A_\phi\cos\theta}{r\sin\theta} \right) \\
&\quad + \hat{\theta}\left( -\frac{\partial A_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r} \right) \\
&\quad + \hat{\phi}\left( \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r} \right)
\end{aligned}$$

$$\boxed{\vec{\nabla}\times \vec{A} = \frac{\hat{r}}{r\sin\theta}\left[ \frac{\partial}{\partial \theta}\left( A_\phi\sin\theta \right) - \frac{\partial A_\theta}{\partial \phi} \right] + \frac{\hat{\theta}}{r\sin\theta}\left[ \frac{\partial A_r}{\partial \phi} - \sin\theta\frac{\partial}{\partial r}\left( rA_\phi \right) \right] + \frac{\hat{\phi}}{r}\left[ \frac{\partial}{\partial r}\left( rA_\theta \right) - \frac{\partial A_r}{\partial \theta} \right]}$$

## Laplacian

The Laplacian is a scalar operator that can be determined from its definition as

$$\nabla^2 u = \vec{\nabla}\cdot\left(\vec{\nabla}u\right) = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \hat{r}\cdot\frac{\partial}{\partial r}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\hat{\theta}}{r}\cdot\frac{\partial}{\partial\theta}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

With the help of the partial derivatives previously obtained, we find

$$\nabla^2 u = \hat{r}\cdot\left(\hat{r}\frac{\partial^2 u}{\partial r^2} - \frac{\hat{\theta}}{r^2}\frac{\partial u}{\partial\theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial\theta\partial r} - \frac{\hat{\phi}}{r^2\sin\theta}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial^2 u}{\partial\phi\partial r}\right)$$

$$+ \frac{\hat{\theta}}{r}\cdot\left(\hat{\theta}\frac{\partial u}{\partial r} + \hat{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\hat{r}}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial\theta^2} - \frac{\hat{\phi}\cos\theta}{r\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial^2 u}{\partial\phi\partial\theta}\right)$$

$$+ \frac{\hat{\phi}}{r\sin\theta}\cdot\left(\hat{\phi}\sin\theta\frac{\partial u}{\partial r} + \hat{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\hat{\phi}\cos\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\hat{r}\sin\theta + \hat{\theta}\cos\theta}{r\sin\theta}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial^2 u}{\partial\phi^2}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\right) + \left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right) + \left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta}\right) + \left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Thus, the Laplacian operator can be written as

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$