# Cylindrical coordinates #rvy

The cylindrical coordinate system extends polar coordinates into 3D by using the standard vertical coordinate $z$. This gives coordinates $(r, \theta, z)$ consisting of:

| coordinate | name | range | definition |
|---|---|---|---|
| $r$ | radius | $0 \le r < \infty$ | distance from the $z$-axis |
| $\theta$ | azimuth | $-\pi < \theta \le \pi$ | angle from the $x$-axis in the $x$–$y$ plane |
| $z$ | height | $-\infty < z < \infty$ | vertical height |

The diagram below shows the cylindrical coordinates of a point $P$. By changing the display options, we can see that the basis vectors are tangent to the corresponding coordinate lines. Changing $\theta$ moves $P$ along the $\theta$ coordinate line in the direction $\hat{e}_\theta$, and similarly for the other coordinates.

Cylindrical coordinates are defined with respect to a set of Cartesian coordinates, and can be converted to and from these coordinates using the atan2 function as follows.

**Conversion between cylindrical and Cartesian coordinates** #rvy-ec

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta & \theta &= \operatorname{atan2}(y, x) \\ z &= z & z &= z \end{aligned}$$

**Derivation** #rvy-ec-d +

The basis vectors are tangent to the coordinate lines and form a right-handed orthonormal basis $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ that depends on the current position $\vec{P}$ as follows. We can write either $\hat{e}_z$ or $\hat{k}$ for the vertical basis vector.

**Cylindrical basis vectors** #rvy-eb

$$\begin{aligned} \hat{e}_r &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \\ \hat{e}_z &= \hat{k} \end{aligned}$$

$$\begin{aligned} \hat{\imath} &= \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \\ \hat{\jmath} &= \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \\ \hat{k} &= \hat{e}_z \end{aligned}$$

**Derivation** #rvy-eb-d +

If the cylindrical coordinates change with time then this causes the cylindrical basis vectors to rotate with the following angular velocity.

**Angular velocity of the cylindrical basis** #rvy-ew

$$\vec{\omega} = \dot{\theta}\,\hat{e}_z$$

**Derivation** #rvy-ew-d +

## Warning! #rvy-ir

We normally write $\vec{r}$ for the position vector of a point, but if we are using cylindrical coordinates $r, \theta, z$ then this is dangerous. This is because $r$ might mean the magnitude of $\vec{r}$ or the radial coordinate, which are different. To avoid this confusion we use $\vec{\rho}$ for the position vector and $r$ for the radial coordinate.

The rotation of the basis vectors caused by changing coordinates gives the time derivatives below.

**Time derivatives of cylindrical basis vectors** #rvy-et

$$\begin{aligned}\dot{\hat{e}}_r &= \dot{\theta}\,\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= -\dot{\theta}\,\hat{e}_r \\ \dot{\hat{e}}_z &= 0\end{aligned}$$

**Derivation** #rvy-et-d +

A point $P$ at a time-varying position $(r, \theta, z)$ has position vector $\vec{\rho}$, velocity $\vec{v} = \dot{\vec{\rho}}$, and acceleration $\vec{a} = \ddot{\vec{\rho}}$ given by the following expressions in cylindrical components.

**Position, velocity, and acceleration in cylindrical components** #rvy-ep

$$\begin{aligned}\vec{\rho} &= r\,\hat{e}_r + z\,\hat{e}_z \\ \vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta + \ddot{z}\,\hat{e}_z\end{aligned}$$

**Derivation** #rvy-ep-d +