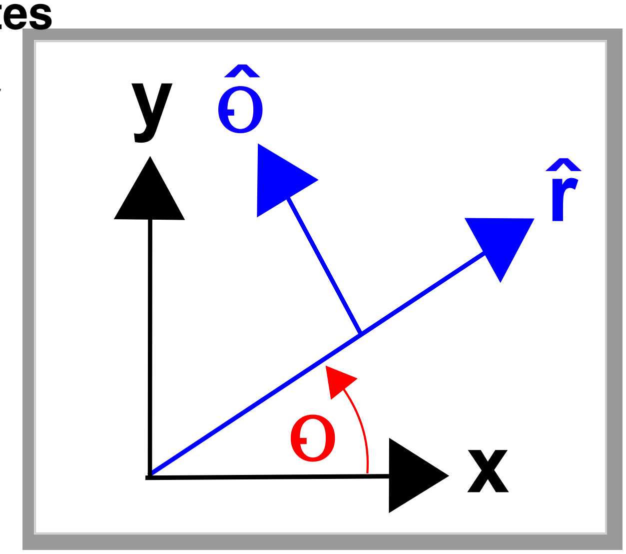
Cylindrical Coordinates

Retrieved from: https://www.continuummechanics.org/cylindricalcoords.html

Introduction

This page covers cylindrical coordinates. The initial part talks about the relationships between position, velocity, and acceleration. The second section quickly reviews the many vector calculus relationships.

Rectangular and Cylindrical Coordinates



Rectangular and cylindrical coordinate systems are related by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

and by

$$r = \sqrt{x^2 + y^2}$$
$$\theta = Tan^{-1}\left(y/x\right)$$
$$z = z$$

Cylindrical coordinates are "polar coordinates plus a z-axis."

Position, Velocity, Acceleration

The position of any point in a cylindrical coordinate system is written as

$$\mathbf{r} = r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}$$

where $\hat{\mathbf{r}} = (\cos\theta, \sin\theta, 0)$. Note that $\hat{\theta}$ is not needed in the specification of $\mathbf{r}$ because θ, and $\hat{\mathbf{r}} = (\cos\theta, \sin\theta, 0)$ change as necessary to describe the position. However, it will appear in the velocity and acceleration equations because

$$\frac{\partial\,\hat{\mathbf{r}}}{\partial\,t} \;=\; \frac{\partial}{\partial\,t}(\cos\theta,\sin\theta,0) \;=\; (-\sin\theta,\cos\theta,0)\frac{\partial\,\theta}{\partial\,t} \;=\; \omega\,\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\,\hat{\boldsymbol{\theta}}}{\partial\,t} \;=\; \frac{\partial}{\partial\,t}(-\sin\theta,\cos\theta,0) \;=\; (-\cos\theta,-\sin\theta,0)\frac{\partial\,\theta}{\partial\,t} \;=\; -\omega\,\hat{\mathbf{r}}$$

and finally $\frac{\partial\,\hat{\mathbf{z}}}{\partial\,t} \;=\; 0$ because $\hat{\mathbf{z}}$ does not change direction.

In summary, identities used here include

$$\omega = \frac{\partial\,\theta}{\partial\,t} \qquad \alpha = \frac{\partial\,\omega}{\partial\,t} \qquad \frac{\partial\,\hat{\mathbf{r}}}{\partial\,t} = \omega\,\hat{\boldsymbol{\theta}} \qquad \frac{\partial\,\hat{\boldsymbol{\theta}}}{\partial\,t} = -\omega\,\hat{\mathbf{r}} \qquad \frac{\partial\,\hat{\mathbf{z}}}{\partial\,t} \;=\; 0$$

Returning to the position equation and differentiating with respect to time gives velocity.

$$\mathbf{v} \;=\; \frac{\partial}{\partial\,t}(r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}) \;=\; (\dot{r}\,\hat{\mathbf{r}} + r\,\omega\,\hat{\boldsymbol{\theta}} + \dot{z}\,\hat{\mathbf{z}})$$

This could also be written as

$$\mathbf{v} = (v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\,\hat{\mathbf{z}})$$

where $v_r = \dot{r}$, $v_\theta = r\,\omega$, and $v_z = \dot{z}$.

Differentiating again to get acceleration...

$$\mathbf{a} = \frac{\partial}{\partial\,t}(\dot{r}\,\hat{\mathbf{r}} + r\,\omega\,\hat{\boldsymbol{\theta}} + \dot{z}\,\hat{\mathbf{z}})$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\omega\,\hat{\boldsymbol{\theta}} + \dot{r}\,\omega\,\hat{\boldsymbol{\theta}} + r\,\alpha\,\hat{\boldsymbol{\theta}} - r\,\omega^2\,\hat{\mathbf{r}} + \ddot{z}\,\hat{\mathbf{z}}$$

$$= (\ddot{r} - r\,\omega^2)\,\hat{\mathbf{r}} + (r\,\alpha + 2\,\dot{r}\,\omega)\,\hat{\boldsymbol{\theta}} + \ddot{z}\,\hat{\mathbf{z}}$$

The $-r\,\omega^2\,\hat{\mathbf{r}}$ term is the centripetal acceleration. Since $\omega = v_\theta/r$, the term can also be written as $-(v_\theta^2/r)\,\hat{\mathbf{r}}$.

The $2\dot{r}\omega\,\hat{\boldsymbol{\theta}}$ term is the Coriolis acceleration. It can also be written as $2\,v_r\,\omega\,\hat{\boldsymbol{\theta}}$ or even as $(2\,v_r\,v_\theta/r)\hat{\boldsymbol{\theta}}$, which stresses the product of v_r and v_θ in the term.