The Rigid Body Equations of Motion

Theorem :→ The rotation angle and instantaneous angular velocity of a rigid body displacement are independent of the choice of origin of the body set of axes.

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Proof :→

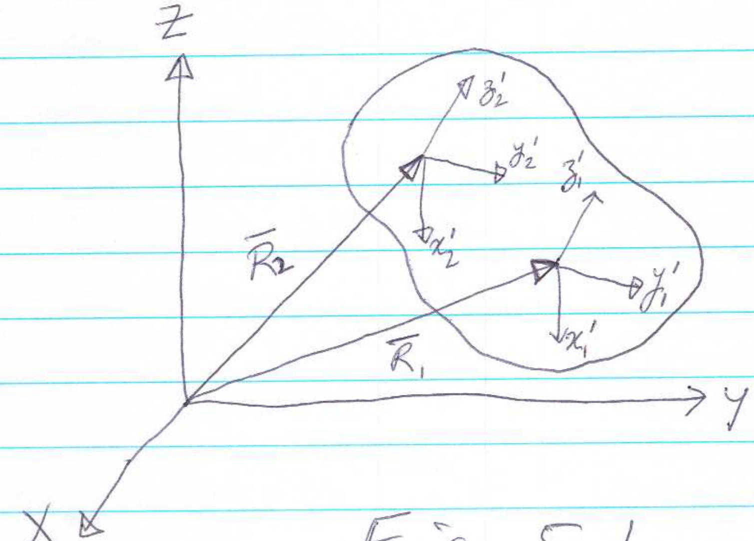


Fig 5.1.

Consider two body axes $(x_i'\ y_i', z_i'), i=1,2$ at $\bar{R}_1$ and $\bar{R}_2$ as shown.

$$\therefore \left(\frac{d\bar{R}_2}{dt}\right)_s = \left(\frac{d\bar{R}_1}{dt}\right)_s + \left(\frac{d\bar{R}}{dt}\right)_s$$

where $\bar{R} \equiv \bar{R}_2 - \bar{R}_1$

$$\left(\frac{d\bar{R}}{dt}\right)_s = \left(\frac{d\bar{R}}{dt}\right)_{B1} + \bar{\omega}_1 \times \bar{R}$$

$\bar{R}$ is a constant vector in any body set of axes

$$\Rightarrow \left(\frac{d\bar{R}}{dt}\right)_{B1} = \left(\frac{d\bar{R}}{dt}\right)_{B2} = \bar{0}$$

$$\Rightarrow \left(\frac{d\bar{R}_2}{dt}\right)_S = \left(\frac{d\bar{R}_1}{dt}\right)_S + \bar{\omega}_1 \times \bar{R} \longrightarrow (5.01)$$

Similarly $\left(\frac{d\bar{R}_1}{dt}\right)_S = \left(\frac{d\bar{R}_2}{dt}\right)_S - \bar{\omega}_2 \times \bar{R} \rightarrow (5.02)$

Eqs. (5.01) and (5.02)

$$\Rightarrow \left(\frac{d}{dt}(\bar{R}_1+\bar{R}_2)\right)_S - \left(\frac{d}{dt}(\bar{R}_1+\bar{R}_2)\right)_S = (\bar{\omega}_1 - \bar{\omega}_2)\times\bar{R}$$

$$\Rightarrow (\bar{\omega}_1 - \bar{\omega}_2)\times\bar{R} = \bar{0}$$

This holds true $\forall\, \bar{R}_1$ and $\bar{R}_2$ points in the rigid body

$\Rightarrow \bar{\omega}_1 = \bar{\omega}_2$, since $\bar{\omega}$ should vary continuously in the body.

Proof done.

The angular momentum $\bar{L}$ is

$$\sum_i m_i \bar{r}_i \times \bar{v}_i$$

$$\bar{v}_i = \left(\frac{d\bar{r}_i}{dt}\right)_S = \left(\frac{d\bar{r}_i}{dt}\right)_B + \bar{\omega} \times \bar{r}_i$$

Let the origin be chosen at the fixed point on the rigid body

$$\Rightarrow \left(\frac{d\bar{r}_i}{dt}\right)_B = \bar{0}$$

$$\Rightarrow \bar{L} = \sum_i m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i)$$

$$= \sum_i m_i \left[\bar{r}_i^2 \bar{\omega} - (\bar{r}_i \cdot \bar{\omega}) \bar{r}_i \right] \rightarrow 5\text{-}3$$

$$\therefore L_x = \sum_i \left[m_i (r_i^2 - x_i^2) \omega_x - m_i x_i y_i \omega_y - m_i x_i z_i \omega_z \right]$$

$$\Rightarrow \bar{L} = \bar{\bar{I}}_M \bar{\omega}, \text{ where } \bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\bar{\bar{I}}_M = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\bar{\bar{I}}_M^T = \bar{\bar{I}}_M \Leftrightarrow \bar{\bar{I}}_M \text{ is symmetric}$$

$\bar{\bar{I}}^M \equiv \bar{\bar{I}}_M \equiv$ moment of inertia matrix or tensor.

$$I^{M}_{xx} = \sum_i m_i (\bar{r}_i^2 - x_i^2) = \int_V \rho(\bar{r})(\bar{r}^2 - x^2)\, dV$$

$$I^{M}_{yy} = \sum_i m_i (\bar{r}_i^2 - y_i^2) = \int_V \rho(\bar{r})(\bar{r}^2 - y^2)\, dV$$

$$I^{M}_{zz} = \sum_i m_i (\bar{r}_i^2 - z_i^2) = \int_V \rho(\bar{r})(\bar{r}^2 - z^2)\, dV$$

$$I^{M}_{xy} = -\sum_i m_i x_i y_i = -\int_V \rho(\bar{r})[xy]\, dV$$

$$I^{M}_{yz} = -\sum_i m_i y_i z_i = -\int_V \rho(\bar{r})[yz]\, dV$$

$$I^{M}_{zx} = -\sum_i m_i z_i x_i = -\int_V \rho(\bar{r})[zx]\, dV$$

The latter expressions are for a continuum rigid body of density $\rho(\bar{r})$ at point $\bar{r}$ and having volume V.

Tensors :→

In a 3-dimensional (3D) Cartesian space a tensor $\overleftrightarrow{T}$ of the N^{TH} rank is a quantity having 3^N components $\overleftrightarrow{T}_{ijk\ldots}$ (with N indices) that transform under an orthogonal transformation $\bar{\bar{A}}$ as

$$T'_{ijk\ldots}(\bar{x}') = \sum_{l,m,n\ldots} A_{il} A_{jm} A_{kn} \ldots T_{lmn\ldots}(\bar{x}),$$

$\boxed{5\text{-}5}$

~~As~~ A tensor of the zeroth rank is a scalar $\therefore$

$$T'(\bar{x}') = T(\bar{x}).$$

A tensor of the first rank is a vector $\therefore$

$$T'_{ij} = \sum_{l,m} A_{il} A_{jm} T_{lm}$$

$$= \sum_{l,m} A_{il} T_{lm} (A^T)_{mj} =$$

$$\Rightarrow \overleftrightarrow{T}' = \bar{\bar{A}}\, \overleftrightarrow{T}\, \bar{\bar{A}}^T$$

But $\bar{\bar{A}}$ is orthogonal $\Rightarrow \bar{\bar{A}}^T = \bar{\bar{A}}^{-1}$

$$\Rightarrow \overleftrightarrow{T}' = \bar{\bar{A}}\, \overleftrightarrow{T}\, \bar{\bar{A}}^{-1}$$

$\Rightarrow$ This transformation of $\overleftrightarrow{T}$ is similar to that of matrix under a rotation $\bar{\bar{A}}$.

$\therefore$ $\overleftrightarrow{T}$ of second rank is a matrix. For this rank we will use $\overleftrightarrow{T} \equiv \bar{\bar{T}}$. Consider now two vectors

$\bar{C}$ and $\bar{D}$

Let $\overline{\overline{T}} = \overline{C}\,\overline{D}^T = \overline{C}_{n\times 1}\,\overline{D}^T_{1\times n} = \overline{\overline{T}}_{n\times n}$.

$\therefore\ T'_{ij} = \sum_{l,m} (a_{il} C_l)(a_{jm} D_m)$ $\boxed{5\text{-}6}$

$= \sum_{l,m} a_{ij} a_{jm} C_l D_m = \sum_{l,m} a_{il} a_{jm} T_{lm}$

$\Rightarrow \overline{\overline{T}}$ is a tensor of second rank, $\overleftrightarrow{T}$.

If $\overline{E} \equiv \overleftrightarrow{T} \cdot \overrightarrow{F}$

$\Leftrightarrow E_i = \sum_j T_{ij} F_j$

If $\overline{E} \equiv \overline{F} \cdot \overleftrightarrow{T} \Leftrightarrow E_i = \sum_j F_j T_{ji}$

If $S \equiv \overline{G} \cdot \overleftrightarrow{T} \cdot \overline{H} = \sum_{ij} G_i T_{ij} H_j$

If $\overleftrightarrow{T} = \overline{A}\,\overline{B}^T \Leftrightarrow T_{ij} = A_i B_j$

the $\overleftrightarrow{T}\,\overline{C} = (\overline{A}\,\overline{B}^T)\,\overline{C} = \overline{A}(\overline{B}^T\overline{C}) = (\overline{B}^T\overline{C})\,\overline{A}$

Note $\overline{F}^T\,\overline{\overline{T}} = \sum_i F_i T_{ij} = E_j$

$\overline{\overline{T}}\,\overline{F} = \sum_j T_{ij} F_j$

$\therefore\ \overline{E}^T = \overline{F}^T\,\overline{\overline{T}}$ is the same as

$\overline{E} = \overline{F} \cdot \overleftrightarrow{T}$

$\therefore$ if $\overleftrightarrow{T} = \bar{A} \cdot \bar{B} \equiv \bar{A}\bar{B}^T$

then $\bar{F} \cdot \overleftrightarrow{T} = \bar{F} \cdot \bar{A} \cdot \bar{B} = (\bar{F}^T \bar{A})\bar{B}^T = (\bar{A}^T \bar{F})\bar{B}^T$

The ~~…~~ Inertia Tensor and The Moment of Inertia, i.e. $\overleftrightarrow{I}$ and I $\boxed{5-7}$

Consider $T = \frac{1}{2}\sum_i m_i \bar{v}_i^2$

Let the origin be at the fixed point on the rigid body

$\therefore \bar{v}_i = \bar{\omega} \times \bar{r}_i$

$$\therefore T = \frac{1}{2}\sum_i m_i \bar{v}_i \cdot (\bar{\omega} \times \bar{r}_i)$$

$$= \frac{1}{2}\bar{\omega} \cdot \sum_i m_i (\bar{r}_i \times \bar{v}_i)$$

$$= \frac{1}{2}\bar{\omega} \cdot \bar{L}$$

But by Eq. (5-3) $\bar{L} = \overleftrightarrow{I} \cdot \bar{\omega}$

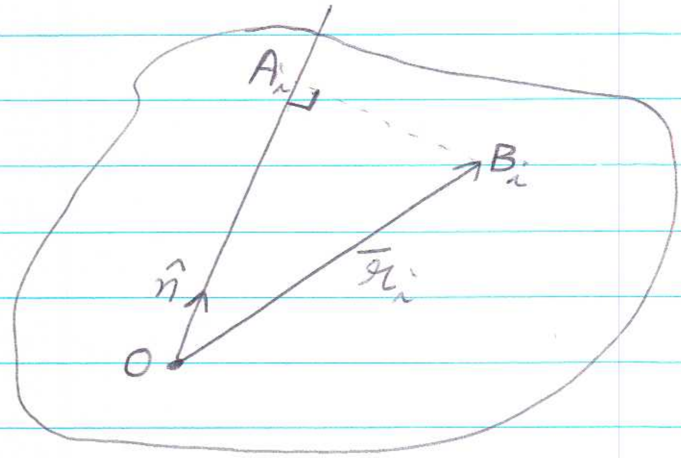
$$\Rightarrow T = \frac{1}{2}\bar{\omega} \cdot \overleftrightarrow{I}_M \cdot \bar{\omega}$$

Let $\bar{\omega} \equiv \omega \hat{n} \Rightarrow T = \frac{\omega^2}{2} \hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n} = \frac{\omega^2}{2} I$

where $I_M \equiv \hat{n} \cdot \overleftrightarrow{I}_M \cdot \hat{n} \equiv \sum_i m_i \left[\bar{r}_i^2 - (\bar{r}_i \cdot \hat{n})^2 \right]$

I_M is called the moment of inertia about the axis of rotation.

Fig 5.2



Note the elementary definition of

$$I = \sum_i m_i \left(\overrightarrow{A_i B_i}\right)^2$$

But $\left(\overrightarrow{A_i B_i}\right)^2 = \bar{r}_i^{\,2} - \left(\overrightarrow{OA_i}\right)^2$

$$= \bar{r}_i^{\,2} - (\bar{r} \cdot \hat{n})^2$$

$$\therefore \quad I = \sum_i m_i \left[\bar{r}_i^{\,2} - (\bar{r}_i \cdot \hat{n})^2\right]$$

same as we derived earlier.
Also note

$$I = \sum_i m_i \left(\overrightarrow{A_i B_i}\right)^2 = \sum_i m_i (\hat{n} \times \bar{r}_i)^2$$

$$= \sum_i \frac{m_i}{\omega^2} (\bar{\omega} \times \bar{r}_i)^2 = \frac{2T}{\omega^2},$$

$\Rightarrow T = \frac{1}{2} I \omega^2$ as earlier.

Parallel axis theorem

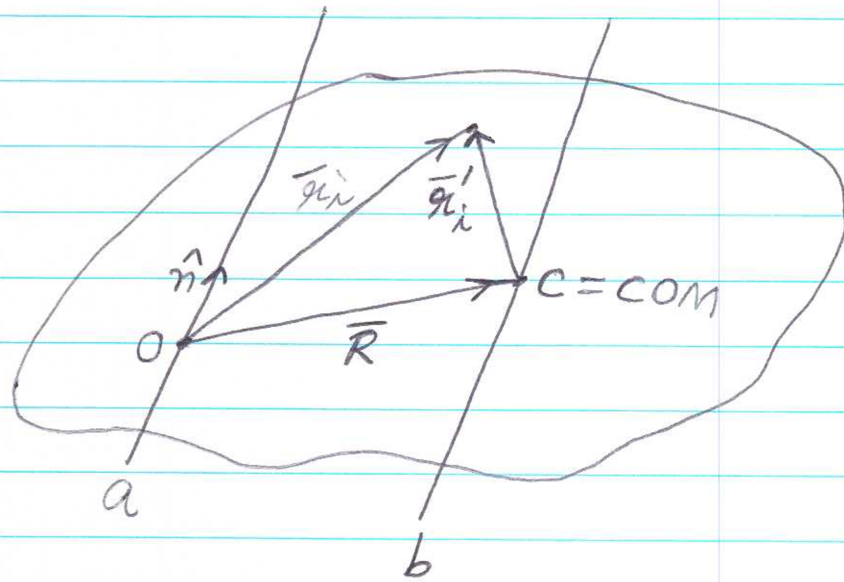


Fig 5.3.

$$I_a = I_b + M(\bar{R}\times\hat{n})^2$$

I_a = moment of inertia with the origin at O and around axis a.

I_b = moment of inertia with the origin at COM and around axis b through COM parallel to b.

From Fig 5.3

$$\bar{r}_i = \bar{R} + \bar{r}_i'$$

$$I_a \equiv \sum_i m_i(\bar{r}_i \times \hat{n})^2 = \sum_i m_i\left[(\bar{R}+\bar{r}_i')\times\hat{n}\right]^2$$

$$= \sum_i m_i(\bar{R}\times\hat{n})^2 + \sum_i m_i(\bar{r}_i'\times\hat{n})^2 + S$$

$\boxed{5\text{-}10}$

$$S = 2\sum_i m_i (\bar{R}\times\hat{n})\cdot(\bar{r}_i'\times\hat{n})$$

$$= 2(\bar{R}\times\hat{n})\cdot\sum_i m_i(\bar{r}_i - \bar{R})\times\hat{n}$$

$$= -2(\bar{R}\times\hat{n})\cdot\left[\hat{n}\times\left[\sum_i m_i\bar{r}_i - M\bar{R}\right]\right]$$

$$= -2(\bar{R}\times\hat{n})\cdot[\hat{n}\times\bar{0}] = 0$$

$$I_b \equiv \sum_i m_i(\bar{r}_i'\times\hat{n})^2$$

$$\Rightarrow I_a = I_b + M(\bar{R}\times\hat{n})^2$$

Theorem proved.

Also $T = \frac{1}{2}\sum_i m_i(\bar{\omega}\times\bar{r}_i)^2$

$$= \frac{1}{2}\sum_i\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} m_i\omega_\alpha\omega_\beta(\delta_{\alpha\beta}r_i^2 - r_{i\alpha}r_{i\beta})$$

$$= \frac{1}{2}\sum_\alpha\sum_\beta I^M_{\alpha\beta}\omega_\alpha\omega_\beta$$

where $I^M_{\alpha\beta} \equiv \sum_i m_i(\delta_{\alpha\beta}r_i^2 - r_{i\alpha}r_{i\beta})$. $\rightarrow$ (5.22)

For continuous bodies

$$I^M_{\alpha\beta} \equiv \int_V \rho(\bar{r})\left[\delta_{\alpha\beta}r^2 - r_\alpha r_\beta\right]dV$$

$\rightarrow$ (5.23)

Prove for a cube of mass M and side a that

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$$I^M_{\alpha\beta} = Ma^2\left[\frac{2}{3}\delta_{\alpha\beta} - \frac{1}{4}(1-\delta_{\alpha\beta})\right]$$

$$= Ma^2\left[\frac{11}{12}\delta_{\alpha\beta} - \frac{1}{4}\right]$$, calculated with the origin at one edge and the 3 sides along $+x, +y, +z$ axes.

Note $I^M_{\alpha\beta}$ depends on the origin and direction of the axes.

Parallel axis theorem for $I_{\alpha\beta}$ the inertia tensor: $\rightarrow$

$$^M I^{(a)}_{\alpha\beta} = {}^M I^{(b)}_{\alpha\beta} + \sum_i \left[\delta_{\alpha\beta} R^2 - R_\alpha R_\beta\right] m_i$$

$$= {}^M I^{(b)}_{\alpha\beta} + M\left(\delta_{\alpha\beta} R^2 - R_\alpha R_\beta\right)$$

Proof: $\rightarrow$ By (5.22) we get

$$^M I^{(a)}_{\alpha\beta} = \sum_i m_i\left(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right)$$

$$= \sum_i m_i\left[\delta_{\alpha\beta}\left(R^2 + r_i'^2 + 2\bar{R}\cdot\bar{r}_i'\right)\right.$$

$$\left. - \left(R_\alpha + r'_{i\alpha}\right)\left(R_\beta + r'_{i\beta}\right)\right]$$

$$= {}^M I^{(b)}_{\alpha\beta} + M\left(\delta_{\alpha\beta} R^2 - R_\alpha R_\beta\right) + S_2$$

where $S_2 = 2\delta_{\alpha\beta}\bar{R}\cdot\sum_i m_i \bar{r}_i'$

$- R_\alpha \sum_i m_i r_{i\beta}' - R_\beta \sum_i m_i r_{i\alpha}'$

$= 0$ because $\sum_i m_i \bar{r}_i' = \bar{0}$, by definition of the center of mass,

$\therefore S_2 = 0$, proof complete.

Perpendicular axis theorem :→
The moment of inertia of a planar object about an axis normal to its plane is equal to the moments of inertia about any two mutually perpendicular axes lying in the plane and passing through the axis.

Prove as an excercise.

The inertia matrix $\bar{\bar{I}}$ or equivalently the inertia tensor $\overleftrightarrow{I}$ depends on the origin of co-ordinates and also on the orientation of the axis.

In general $\bar{L} = \overleftrightarrow{I}_M \cdot \bar{\omega} \Rightarrow \bar{L}$ is not parallel to $\bar{\omega}$. At best $\overleftrightarrow{I}_M$ can be diagonal $\Rightarrow L_i = I_i^M \omega_i$, $\forall i = 1, 2, 3$.

The axes in which $\overleftrightarrow{I}_M$ is diagonal are called the principal axes of a rigid body. The diagonal elements of $\overleftrightarrow{I}_M$ in this axes are called the principal moments of ~~m~~ inertia. These are usually associated with the symmetry of mass distribution in the rigid body.

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Diagonalising $\overline{\overline{I}}_M$: →

Let $\overline{\overline{I}}_M \bar{a}_k = \lambda_k \bar{a}_k \quad , \forall k = 1, 2, 3.$

$\bar{a}_k \equiv$ Eigenvectors of $\overline{\overline{I}}_M$ and
$\lambda_k \equiv$ Eigenvalues of $\overline{\overline{I}}_M$

λ_k are also called the principal moments of inertia.

Let $\hat{e}_k$ be unit vectors $\ni e_{ki} = \delta_{ik}$, $k = 1, 2, 3.$

Let $\overline{\overline{A}} = \sum_k \hat{e}_k \bar{a}_k^T$

$\therefore \quad \overline{\overline{I}}_M \overline{\overline{A}} = \overline{\overline{\lambda}}\, \overline{\overline{A}}$ where

$\overline{\overline{\lambda}}$ is a matrix $\ni \lambda_{ij} = \lambda_i \delta_{ij}$.

$\therefore \quad \overline{\overline{A}}^{\dagger} \overline{\overline{I}}_M \overline{\overline{A}} = A^{\dagger} \overline{\overline{\lambda}}\, \overline{\overline{A}}$

By definition of $\overline{\overline{I}}_M$ we get $\overline{\overline{I}}_M^{\dagger} = \overline{\overline{I}}_M$

$\Rightarrow (\overline{\overline{A}}^{\dagger} \overline{\overline{I}}_M \overline{\overline{A}})^{\dagger} = \overline{\overline{A}}^{\dagger} \overline{\overline{I}}_M \overline{\overline{A}}$

$$\Rightarrow \quad \bar{\bar{A}}^{\dagger}\bar{\bar{\lambda}}^{\dagger}\bar{\bar{A}} = \bar{\bar{A}}^{\dagger}\bar{\bar{\lambda}}\bar{\bar{A}}$$

$\Rightarrow \bar{\bar{\lambda}}^{\dagger} = \bar{\bar{\lambda}} \Leftrightarrow$ principal moments of inertia are real.

$\Rightarrow$ we can choose $\bar{\bar{A}}^{*} = \bar{\bar{A}}$

$\Rightarrow \bar{a}_k^{*} = \bar{a}_k \Rightarrow$ eigenvectors can be chosen to be real. We can also normalize them $\ni \bar{a}_k^{T}\bar{a}_j = \delta_{jk}$

$\Rightarrow \bar{\bar{A}}^{T}\bar{\bar{A}} = \bar{\bar{I}} \Leftrightarrow \bar{\bar{A}}$ is a rotation matrix. Also

$$\bar{\bar{I}}_M \bar{a}_k = \lambda_k \bar{a}_k \quad \text{with } |\bar{a}_k|^2 = 1$$

$\Rightarrow \bar{a}_k$ is a unit vector. If we choose

$$\bar{\omega} = \omega \bar{a}_k \quad \text{then}$$

$$= \bar{\bar{I}}_M \bar{\omega} = \omega \lambda_k \bar{a}_k$$

$$\Rightarrow \quad L_k = \lambda_k \omega \quad \text{where } L_k = \bar{L} \cdot \bar{a}_k$$

The axes along $\{\bar{a}_k\}$ are called the principal axes of a rigid body.