

Mechanics is the study of particles & systems of particles.

Classical - denotes pre-quantum mechanics. Includes special and general relativity.

$\vec{r}$  = position vector

$\vec{v}$  = velocity

$$\vec{v} \equiv \frac{d\vec{r}}{dt}, \quad \vec{p} \equiv m\vec{v}$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \quad \rightarrow \quad (1.3)$$

If  $m = \text{constant}$  then

$$\vec{F} = m\vec{a} \quad \text{where} \quad \vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\therefore \vec{a} \equiv \dot{\vec{v}} \equiv \ddot{\vec{r}}.$$

When  $\textcircled{1.3}$  holds that frame of reference is an inertial system.

$$\textcircled{1} \quad \bar{F} = \dot{\bar{p}}$$

$$\begin{aligned} \bar{r} \times \bar{F} &= \bar{r} \times \dot{\bar{p}} = \bar{r} \times \frac{d\bar{v}}{dt} m \\ &\quad + \bar{v} \times m \bar{v} \\ &= \frac{d}{dt} (\bar{r} \times \bar{v} m) = \frac{d\bar{L}}{dt} \end{aligned}$$

where  $\bar{L} \equiv m \bar{r} \times \bar{v}$

$$\bar{N} \equiv \bar{r} \times \bar{F} = \frac{d\bar{L}}{dt} = \dot{\bar{L}}$$

$\textcircled{2.1}$   
If  $\bar{N} = \bar{0}$  then  $\bar{L}$  is conserved  
Conservation of angular momentum.  
If  $\bar{F} = \bar{0}$  then  $\bar{p}$  is constant. Conservation of linear momentum.

$$W_{12} \equiv \int_1^2 \bar{F} \cdot d\bar{s}$$

$$= m \int_1^2 \left( \frac{d\bar{v}}{dt} \right) \cdot (\bar{v} dt) = \frac{m}{2} \int_1^2 \frac{d(\bar{v}^2)}{dt} dt$$

$$= \frac{m}{2} (\bar{v}_2^2 - \bar{v}_1^2)$$

$$\therefore W_{12} = T_2 - T_1$$

A force is conservative if

$$\oint \bar{F} \cdot d\bar{s} = 0$$

By vector analysis then

$$\bar{F} = -\bar{\nabla} V(\bar{r})$$

$$\therefore W_{12} = - \int_1^2 [\bar{\nabla} V(\bar{r})] \cdot d\bar{s}$$

$$= -V_2(\bar{r}_2) + V_1(\bar{r}_1) = -V(\bar{r}_2) + V(\bar{r}_1)$$

$$= V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

## ⇒ Energy Conservation.

If the forces acting on a particle are conservative then the total energy defined as  $T+V$  is conserved.

Mechanics of a system of particles

$$\left( \sum_j \vec{F}_{ji} \right) + F_i^e = \dot{\vec{p}}_i$$

$$\sum_i \dot{\vec{p}}_i = \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i F_i^e + \sum_{\substack{j \\ i \neq j}} F_{ji}$$

Define  $\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$

$$\therefore \sum_i \dot{\vec{p}}_i = M \frac{d^2 \vec{R}}{dt^2} = \sum_i F_i^e$$

where we assumed  $\vec{F}_{ij} = -\vec{F}_{ji}$ ,  $\forall i, j$

⇒ Irrelevance of internal forces

$\bar{F}_{ij}$  for the center of mass (COM) motion.

$$\sum_i m_i \bar{v}_i = M \frac{d\bar{R}}{dt} = \bar{P}$$

If  $\bar{F}_e = \bar{0} \Rightarrow \bar{P} = \text{constant}$

$\Rightarrow$  Conservation of total linear momentum.

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \frac{d}{dt} (\bar{r}_i \times \bar{p}_i) = \frac{d\bar{L}}{dt} = \dot{\bar{L}}$$

where  $\bar{L} \equiv \sum_i \bar{r}_i \times \bar{p}_i$

$$\dot{\bar{L}} = \sum_i \bar{r}_i \times \bar{F}_i^e + \sum_{\substack{i,j \\ i \neq j}} (\bar{r}_i \times \bar{F}_{ji})$$

Consider  $\bar{r}_i \times \bar{F}_{ji} + \bar{r}_j \times \bar{F}_{ij}$

$$= (\bar{r}_i - \bar{r}_j) \cdot \bar{F}_{ji} \equiv \bar{r}_{ij} \times \bar{F}_{ji}$$

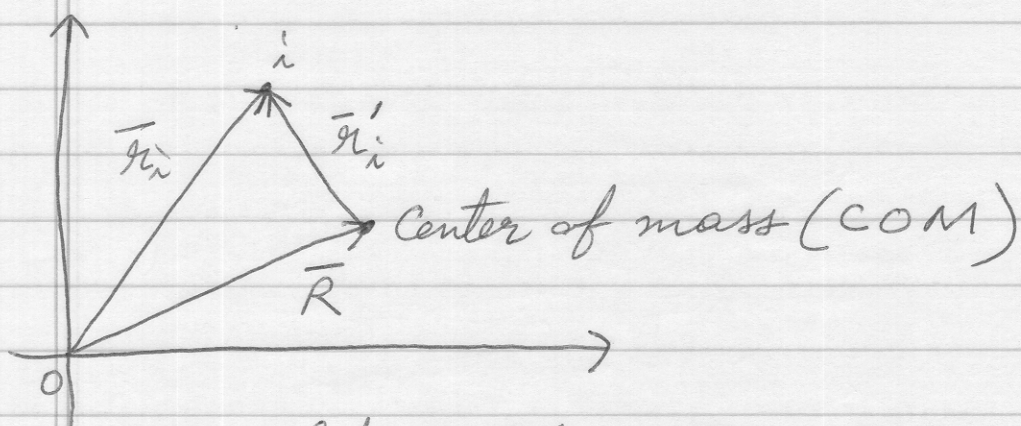
If  $\bar{F}_{ji} \parallel \bar{r}_{ij}$  then

$$\bar{r}_{ij} \times \bar{F}_{ji} = \bar{0}$$

$$\therefore \dot{\bar{L}} = \sum_i \bar{r}_i \times \bar{F}_i^e = \bar{N}^e$$

$\Rightarrow$  Conservation of angular momentum.

Total angular momentum  $\bar{L} = \text{constant}$   
iff  $\bar{N}^e = \bar{0}$ .



$$\text{Let } \bar{r}'_i = \bar{r}_i - \bar{R}$$

$$\bar{v}'_i = \bar{v}_i - \bar{v}, \text{ where } \bar{v} \equiv \dot{\bar{R}}$$

$$\bar{v}'_i \equiv \dot{\bar{r}}'_i$$

$$\begin{aligned} \bar{L} &\equiv \sum_i \bar{r}_i \times \bar{p}_i = \sum_i (\bar{r}'_i + \bar{R}) \times (\bar{v} + \bar{v}'_i) m_i \\ &= \sum_i \left[ \bar{R} \times m_i \bar{v} + \bar{r}'_i \times m_i \bar{v}'_i \right] \\ &\quad + \left( \sum_i m_i \bar{r}'_i \right) \times \bar{v} + \bar{R} \times \frac{d}{dt} \sum_i m_i \bar{r}'_i \end{aligned}$$

$$\therefore \bar{L} \equiv \bar{R} \times M\bar{V} + \sum_i \bar{r}'_i \times \bar{p}'_i$$

⇒ Total angular momentum about an arbitrary origin O

= Angular momentum of COM about O,

+ angular momentum about COM.

Work done by a force.

$$W_{12}^S = \sum_i \int_1^2 \bar{F}_i \cdot d\bar{s}_i = \sum_i \int_1^2 (\bar{F}_i^e + \bar{F}_{ji}) \cdot d\bar{s}_i \rightarrow (1.29)$$

$$= \sum_i \int_1^2 m_i \bar{v}_i \cdot d\bar{v}_i = \sum_i \int_1^2 d\left(\frac{1}{2} m_i \bar{v}_i^2\right)$$

$$= T_2^S - T_1^S$$

where  $T^S \equiv \sum_i \frac{1}{2} m_i \bar{v}_i^2$

$$= \sum_i \frac{1}{2} m_i (\bar{v}'_i + \bar{V})^2 = \frac{M}{2} \bar{V}^2 + \frac{1}{2} \sum_i m_i \bar{v}'_i^2$$

because  $\sum_i m_i \bar{v}_i' \cdot \bar{v}$

$$= \bar{v} \cdot \sum_i m_i \frac{d(\bar{r}_i' m_i)}{dt} = \bar{v} \cdot \frac{d}{dt} \left( \sum_i m_i \bar{r}_i' \right)$$

$$\sum_i m_i \bar{r}_i' = \sum_i m_i (\bar{r}_i - \bar{R})$$

$$= M \bar{R} - M \bar{R} = \bar{0}$$

$$\therefore T^S = \frac{1}{2} M \bar{v}^2 + \frac{1}{2} \sum_i m_i \bar{v}_i'^2$$

$\Rightarrow$  Kinetic ~~over~~ energy of a system of particles  $T^S =$  Kinetic energy of its COM + Kinetic energy about COM.

Consider conservative form for  $\bar{F}_i^e$

$$\Rightarrow \bar{F}_i^e = -\bar{\nabla}_i V_i$$

$$\therefore \sum_i \int_1^2 \bar{F}_i^e \cdot d\bar{s}_i = - \int_1^2 \sum_i (\bar{\nabla}_i V_i) \cdot d\bar{s}_i = \sum_i V_i \Big|_1^2$$



If  $\bar{F}_{ij}$  are conservative then

$$\bar{F}_{ji} = -\bar{\nabla}_i V_{ij}$$

Strong form of Newton's 3<sup>rd</sup> law

$$\Rightarrow \bar{F}_{ji} = -\bar{\nabla}_i V_{ij} = \bar{\nabla}_j V_{ij} = \bar{F}_{ij}$$

$$\Rightarrow V_{ij}(\bar{x}_i, \bar{x}_j) = V_{ij}(|\bar{x}_i - \bar{x}_j|)$$

$$\bar{\nabla} V_{ij}(|\bar{x}|) = \bar{x} f, \text{ } f \text{ is a scalar.}$$

Consider now pairs of forces  $\bar{F}_{ij}$  and  $\bar{F}_{ji}$

$$\therefore \int_1^2 [\bar{F}_{ji} \cdot d\bar{s}_i + \bar{F}_{ij} \cdot d\bar{s}_j]$$

$$= \int_1^2 [-(\bar{\nabla}_i V_{ij}) \cdot d\bar{s}_i + (-\bar{\nabla}_j V_{ij}) \cdot d\bar{s}_j]$$

$$= \int_1^2 (\bar{\nabla}_i V_{ij}) \cdot (d\bar{s}_j - d\bar{s}_i)$$

$$d\bar{s}_j - d\bar{s}_i = -d\bar{x}_{ij} \equiv -(d\bar{x}_i - d\bar{x}_j)$$

Also  $\bar{\nabla}_i V_{ij} = \bar{\nabla}_{ij} V_{ij}$

$$\begin{aligned}
 \therefore \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i \\
 &= -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \int_1^2 (\vec{\nabla}_i \cdot \vec{V}_{ij}) \cdot d\vec{r}_{ij} \\
 &= -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij} \Big|_1^2
 \end{aligned}$$

Define

$$V^S = \sum_i V_i + \frac{1}{2} \sum'_{ij} V_{ij}$$

$$\therefore W_{12}^S = T_2^S - T_1^S$$

$$\text{Also } W_{12}^S = \sum_i V_i \Big|_2^1 + \frac{1}{2} \sum'_{ij} V_{ij} \Big|_2^1$$

$$\therefore W_{12}^S = V_1^S - V_2^S$$

$$\Rightarrow T_2^S - T_1^S = V_1^S - V_2^S$$

$$V_1^S + T_1^S = V_2^S + T_2^S$$

$\Rightarrow$  Conservation of Energy for a system of particles