(a) Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.

(b) Calculate the period of radial oscillations for slight disturbances from this circular orbit.

(c) Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity?

21. Show that the motion of a particle in the potential field

$$V(r) = -\frac{k}{r} + \frac{h}{r^2}$$

is the same as that of the motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of force.

For negative total energy, show that if the additional potential term is very small compared to the Kepler potential, then the angular speed of precession of the elliptical orbit is

$$\dot{\Omega} = \frac{2\pi m h}{l^2 \tau}.$$

The perihelion of Mercury is observed to precess (after correction for known planetary perturbations) at the rate of about 40″ of arc per century. Show that this precession could be accounted for classically if the dimensionless quantity

$$\eta = \frac{h}{ka}$$

(which is a measure of the perturbing inverse-square potential relative to the gravitational potential) were as small as 7×10^{-8}. (The eccentricity of Mercury's orbit is 0.206, and its period is 0.24 year.)

22. The additional term in the potential behaving as r^{-2} in Exercise 21 looks very much like the centrifugal barrier term in the equivalent one-dimensional potential. Why is it then that the additional force term causes a precession of the orbit, while an addition to the barrier, through a change in l, does not?

23. Evaluate approximately the ratio of mass of the Sun to that of Earth, using only the lengths of the year and of the lunar month (27.3 days), and the mean radii of Earth's orbit (1.49×10^8 km) and of the Moon's orbit (3.8×10^5 km).

24. Show that for elliptical motion in a gravitational field the radial speed can be written as

$$\dot{r} = \frac{\omega a}{r}\sqrt{a^2 e^2 - (r-a)^2}.$$

Introduce the eccentric anomaly variable ψ in place of r and show that the resulting differential equation in ψ can be integrated immediately to give Kepler's equation.

25. If the eccentricity e is small, Kepler's equation for the eccentric anomaly ψ as a function of ωt, Eq. (3.76), is easily solved on a computer by an iterative technique that treats the $e \sin\psi$ term as of lower order than ψ. Denoting ψ_n by the nth iterative