15. Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi,$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}.$$

16. Show that the Euler parameter e_0 has the equation of motion

$$-2\dot{e}_0 = e_1\omega_{x'} + e_2\omega_{y'} + e_3\omega_{z'},$$

where the prime denotes the body set of axes. Find the corresponding equations for the other three Euler parameters and for the complex Cayley–Klein parameters α and β.

17. Verify directly that the matrix generators of infinitesimal rotation, $\mathbf{M}_i$, as given by Eq. (4.79) obey the commutation relations

$$[\mathbf{M}_i, \mathbf{M}_j] = \epsilon_{ijk}\mathbf{M}_k.$$

18. (a) Find the vector equation describing the reflection of **r** in a plane whose unit normal is **n**.

(b) Show that if l_i, $i = 1, 2, 3$, are the direction cosines of **n**, then the matrix of transformation has the elements

$$A_{ij} = \delta_{ij} - 2l_i l_j,$$

and verify that **A** is an improper orthogonal matrix.

19. Figures 4.9 and 4.10 show that the order of finite rotations leads to different results. Use the notation that $\mathbf{A}(\alpha, \mathbf{1}_n)$ where **A** is a rotation in the direction of $\mathbf{1}_n$ through an angle α. Let n_1 and n_2 be two orthogonal directions.

(a) If **x** is the position vector of a point on a rigid body, which is then rotated by an angle θ around the origin, show that the new value of **x** is

$$\mathbf{x}' = (\mathbf{1}_n \cdot \mathbf{x})\mathbf{1}_n + [\mathbf{x} - \mathbf{1}_n(\mathbf{1}_n \cdot \mathbf{x})]\cos\theta - \mathbf{1}_n \times \mathbf{x}\sin\theta.$$

From this, obtain the formula for $\mathbf{A}(\pi/2, \mathbf{1}_n)$ and derive the two rotations in the figures.

(b) Discuss these two rotations. [*Hint:* The answer will involve a rotation by the angle $\frac{2}{3}\pi$ in a direction $(1/\sqrt{3})(1, 1, 1)$.]

20. Express the "rolling" constraint of a sphere on a plane surface in terms of the Euler angles. Show that the conditions are nonintegrable and that the constraint is therefore nonholonomic.

EXERCISES

21. A particle is thrown up vertically with initial speed v_0, reaches a maximum height and falls back to ground. Show that the Coriolis deflection when it again reaches the ground is opposite in direction, and four times greater in magnitude, than the Coriolis deflection when it is dropped at rest from the same maximum height.