a hyperbolic orbit intersecting Earth's surface. Show how $v'$ and $\phi'$ can be determined from $v$ and $\phi$ in terms of known constants.

16. Prove that in a Kepler elliptic orbit with small eccentricity $e$ the angular motion of a particle as viewed from the *empty* focus of the ellipse is uniform (the empty focus is the focus that is *not* the center of attraction) to first order in $e$. It is this theorem that enables the Ptolemaic picture of planetary motion to be a reasonably accurate approximation. On this picture the Sun is assumed to move uniformly on a circle whose center is shifted from Earth by a distance called the *equant*. If the equant is taken as the distance between the two foci of the correct elliptical orbit, then the angular motion is thus described by the Ptolemaic picture accurately to first order in $e$.

17. One classic theme in science fiction is a twin planet ("Planet X") to Earth that is identical in mass, energy, and momentum but is located on the orbit 90° out of phase with Earth so that it is hidden from the Sun. However, because of the elliptical nature of the orbit, it is not always completely hidden. Assume this twin planet is in the same Keplerian orbit as Earth in such a manner than it is in aphelion when Earth is in perihelion. Calculate to first order in the eccentricity $e$ the maximum angular separation of the twin and the Sun as viewed from the Earth. Could such a twin be visible from Earth? Suppose the twin planet is in an elliptical orbit having the same size and shape as that of Earth, but rotated 180° from Earth's orbit, so that Earth and the twin are in perihelion at the same time. Repeat your calculation and compare the visibility in the two situations.

18. At perigee of an elliptic gravitational orbit a particle experiences an impulse $S$ (cf. Exercise 11, Chapter 2) in the radial direction, sending the particle into another elliptic orbit. Determine the new semimajor axis, eccentricity, and orientation in terms of the old.

19. A particle moves in a force field described by

$$F(r) = -\frac{k}{r^2} \exp\left(-\frac{r}{a}\right),$$

where $k$ and $a$ are positive.

(a) Write the equations of motion and reduce them to the equivalent one-dimensional problem. Use the effective potential to discuss the qualitative nature of the orbits for different values of the energy and the angular momentum.

(b) Show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\rho/a$ per revolution, where $\rho$ is the radius of the circular orbit.

20. A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional force

$$\mathbf{F} = -mC\mathbf{r},$$

where $m$ is the mass of the planet, $C$ is a constant proportional to the gravitational constant and the density of the dust, and $\mathbf{r}$ is the radius vector from the Sun to the planet (both considered as points). This additional force is very small compared to the direct Sun–planet gravitational force.

(a) Calculate the period for a circular orbit of radius $r_0$ of the planet in this combined field.

(b) Calculate the period of radial oscillations for slight disturbances from this circular orbit.

(c) Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity?

**21.** Show that the motion of a particle in the potential field

$$V(r) = -\frac{k}{r} + \frac{h}{r^2}$$

is the same as that of the motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of force.

For negative total energy, show that if the additional potential term is very small compared to the Kepler potential, then the angular speed of precession of the elliptical orbit is

$$\dot{\Omega} = \frac{2\pi m h}{l^2 \tau}.$$

The perihelion of Mercury is observed to precess (after correction for known planetary perturbations) at the rate of about 40″ of arc per century. Show that this precession could be accounted for classically if the dimensionless quantity

$$\eta = \frac{h}{ka}$$

(which is a measure of the perturbing inverse-square potential relative to the gravitational potential) were as small as $7 \times 10^{-8}$. (The eccentricity of Mercury's orbit is 0.206, and its period is 0.24 year.)

**22.** The additional term in the potential behaving as $r^{-2}$ in Exercise 21 looks very much like the centrifugal barrier term in the equivalent one-dimensional potential. Why is it then that the additional force term causes a precession of the orbit, while an addition to the barrier, through a change in $l$, does not?

**23.** Evaluate approximately the ratio of mass of the Sun to that of Earth, using only the lengths of the year and of the lunar month (27.3 days), and the mean radii of Earth's orbit ($1.49 \times 10^8$ km) and of the Moon's orbit ($3.8 \times 10^5$ km).

**24.** Show that for elliptical motion in a gravitational field the radial speed can be written as

$$\dot{r} = \frac{\omega a}{r}\sqrt{a^2 e^2 - (r-a)^2}.$$

Introduce the eccentric anomaly variable $\psi$ in place of $r$ and show that the resulting differential equation in $\psi$ can be integrated immediately to give Kepler's equation.

**25.** If the eccentricity $e$ is small, Kepler's equation for the eccentric anomaly $\psi$ as a function of $\omega t$, Eq. (3.76), is easily solved on a computer by an iterative technique that treats the $e \sin \psi$ term as of lower order than $\psi$. Denoting $\psi_n$ by the $n$th iterative