9. A chain or rope of indefinite length passes freely over pulleys at heights y_1 and y_2 above the plane surface of Earth, with a horizontal distance $x_2 - x_1$ between them. If the chain or rope has a uniform linear mass density, show that the problem of finding the curve assumed between the pulleys is identical with that of the problem of minimum surface of revolution. (The transition to the Goldschmidt solution as the heights y_1 and y_2 are changed makes for a striking lecture demonstration. See Exercise 8.)

10. Suppose it is known experimentally that a particle fell a given distance y_0 in a time $t_0 = \sqrt{2y_0/g}$, but the times of fall for distances other than y_0 is not known. Suppose further that the Lagrangian for the problem is known, but that instead of solving the equation of motion for y as a function of t, it is guessed that the functional form is

$$y = at + bt^2.$$

If the constants a and b are adjusted always so that the time to fall y_0 is correctly given by t_0, show directly that the integral

$$\int_0^{t_0} L\,dt$$

is an extremum for real values of the coefficients only when $a = 0$ and $b = g/2$.

11. When two billiard balls collide, the instantaneous forces between them are very large but act only in an infinitesimal time Δt, in such a manner that the quantity

$$\int_{\Delta t} F\,dt$$

remains finite. Such forces are described as *impulsive* forces, and the integral over Δt is known as the *impulse* of the force. Show that if impulsive forces are present Lagrange's equations may be transformed into

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)_f - \left(\frac{\partial L}{\partial \dot{q}_j}\right)_i = S_j,$$

where the subscripts i and f refer to the state of the system before and after the impulse, S_j is the impulse of the generalized impulsive force corresponding to q_j, and L is the Lagrangian including all the nonimpulsive forces.

12. The term *generalized mechanics* has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of q_i higher than the first. Problems for which $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$ have been referred to as "jerky" mechanics. Such equations of motion have interesting applications in chaos theory (cf. Chapter 11). By applying the methods of the calculus of variations, show that if there is a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$, and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then the corresponding Euler–Lagrange equations are

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n.$$

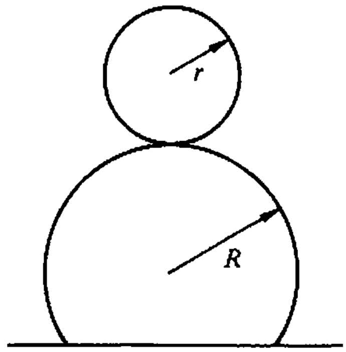
Apply this result to the Lagrangian

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2.$$

Do you recognize the equations of motion?

13. A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrange's undetermined multipliers and Lagrange's equations. Find the height at which the particle falls off.

14. A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange muliplers to find the point at which the hoop falls off the cylinder.



15. A form of the Wheatstone impedance bridge has, in addition to the usual four resistances, an inductance in one arm and a capacitance in the opposite arm. Set up L and $\mathcal{F}$ for the unbalanced bridge, with the charges in the elements as coordinates. Using the Kirchhoff junction conditions as constraints on the currents, obtain the Lagrange equations of motion, and show that eliminating the λ's reduces these to the usual network equations.

16. In certain situations, particularly one-dimensional systems, it is possible to incorporate frictional effects without introducing the dissipation function. As an example, find the equations of motion for the Lagrangian

$$L = e^{\gamma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right).$$

How would you describe the system? Are there any constants of motion? Suppose a point transformation is made of the form

$$s = e^{\gamma t}q.$$

What is the effective Lagrangian in terms of s? Find the equation of motion for s. What do these results say about the conserved quantities for the system?

17. It sometimes occurs that the generalized coordinates appear separately in the kinetic energy and the potential energy in such a manner that T and V may be written in the form

$$T = \sum_i f_i(q_i)\dot{q}_i^2 \qquad \text{and} \qquad V = \sum_i V_i(q_i).$$