Solution to 10(a)

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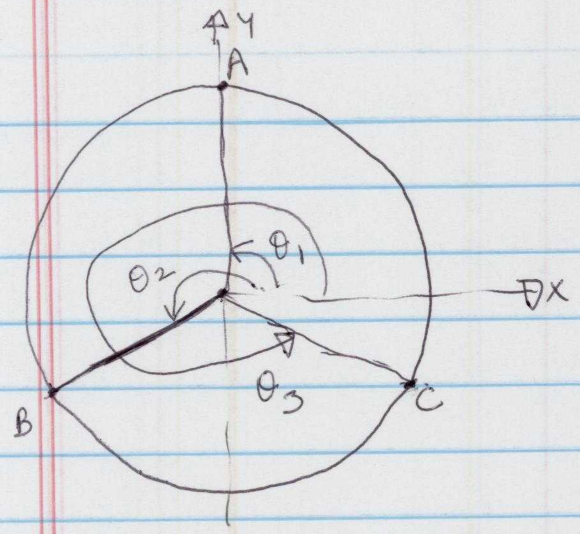

Consider the Fig shown on the left. Let ~~r~~ r be the radius

$$L = T - V \equiv \frac{\dot{\bar{q}}^{T}\,\bar{\bar{T}}\,\dot{\bar{q}} - \bar{q}^{T}\bar{\bar{V}}\,\bar{q}}{2}$$

$$T = \frac{mr^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

$$V = \frac{k}{2}\left[(\theta_1-\theta_2)^2 + (\theta_2-\theta_3)^2 + (\theta_3-\theta_1)^2\right]$$

$$= \frac{k}{2}\left[2\left(\theta_1^2+\theta_2^2+\theta_3^2\right) - 2\left(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1\right)\right]$$

$$\therefore \quad \bar{\bar{T}} = mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = mr^2\,\bar{\bar{I}}$$

$$\bar{\bar{V}} = k \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\det\left(\bar{\bar{V}} - \omega^2\bar{\bar{T}}\right) = 0$$

$$\Rightarrow \begin{vmatrix} 2k - mr^2\omega^2 & -k & -k \\ -k & 2k - mr^2\omega^2 & -k \\ -k & -k & 2k - mr^2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} mr^2\omega^2 - 2k & k & k \\ k & mr^2\omega^2 - 2k & k \\ k & k & mr^2\omega^2 - 2k \end{vmatrix} = 0$$

Let $\omega'^2 \equiv mr^2\omega^2$

$$\Rightarrow (\omega'^2 - 2k)\left[(\omega'^2-2k)^2 - k^2\right]$$

$$- k\left[(\omega'^2-2k)k - k^2\right]$$

$$+ k\left[k^2 - (\omega'^2-2k)k\right] = 0$$

$$\Rightarrow (\omega'^2-2k)^3 - 3k^2(\omega'^2-2k) + 2k^3 = 0$$

$$\Rightarrow \omega'^6 - 6k\omega'^4 + 9k^2\omega'^2 = 0$$

$$\Rightarrow \omega'^2 = 0 \Rightarrow \omega = 0 \quad \text{or}$$

$$\omega'^4 - 6k\omega^2 + 9k^2 = 0$$

$$\Rightarrow \omega'^2 = \frac{6k \pm \sqrt{36k^2 - 36k^2}}{2}$$

$$\Rightarrow \omega' = \pm\sqrt{3k} \Rightarrow \omega = \frac{\pm\sqrt{3k}}{mr^2}$$

Case (i) $\omega = 0$

$$\Rightarrow -2a_1 + a_2 + a_3 = 0$$

$$a_1 - 2a_2 + a_3 = 0$$

$$\Rightarrow -2a_1 + 4a_2 - 2a_3 = 0$$

$$\Rightarrow \cancel{5a_2 - a_3} \quad a_2 + a_3 = 2a_1$$

$$2a_2 - a_3 = a_1$$

$$\Rightarrow \quad 3a_2 = 3a_3 \Rightarrow a_2 = a_3$$

$$\Rightarrow \quad a_1 = \frac{2a_2}{2} = a_2$$

$$\Rightarrow \quad a_1 = a_2 = a_3$$

$$\bar{a}^\dagger \bar{\bar{T}} \bar{a} = 1 \Rightarrow \bar{a} = \frac{1}{(\sqrt{3})r\sqrt{m}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\omega' = +\sqrt{3k} \Rightarrow \omega = +\sqrt{\frac{3k}{mr^2}}$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0 \Rightarrow -a_3 = a_1 + a_2$$

$$\Rightarrow \bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ -a_1 - a_2 \end{bmatrix}$$

Case (iii) $\omega' = -\sqrt{3k} \Rightarrow \omega = -\sqrt{\frac{3k}{mr^2}}$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0 \Rightarrow a_3 = -(a_1 + a_2)$$

Note case (ii) and (iii) are degenerate
Also $\omega = 0$ solution gives $a_1 = a_2 = a_3$
$\Rightarrow$ the whole system just rotates.
If we do not allow any net angular momentum this term will not show up.
For now we need to get a_1 & a_2 in the cases (ii) and (iii).

We need all solutions to be orthogonal,

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So let $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ -a_1 - a_2 \end{bmatrix} = 0$

This is already satisfied. We choose arbitrarily $a_1 = 1 = a_2$ for our first vector in cases (ii) & (iii)

$$\Rightarrow \bar{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\bar{b}^{\dagger} \bar{\bar{T}} \bar{b} = 1 \quad \Rightarrow \bar{b} = \frac{1}{r\sqrt{6m}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Now we need $\bar{c} \ni \bar{c} \cdot \bar{b} = 0$

$$\therefore \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ -a_1 - a_2 \end{bmatrix} = a_1 + a_2 + 2a_1 + 2a_2 = 0$$

$$\Rightarrow a_2 = -a_1$$

$$\Rightarrow \bar{c} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{c}^{\dagger} \bar{\bar{T}} \bar{c} = 1 \Rightarrow \bar{c} = \frac{1}{r\sqrt{2m}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\therefore \bar{\bar{A}} = \left(\frac{1}{r\sqrt{m}\sqrt{6}} \right) \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix}$$

Note that case (i) $\omega = 0$ $\nexists$ cannot do small oscillation solution. E6-10

Solving Lagrange's eqns we find

$$mr^2 \frac{d^2}{dt^2}(\theta_1 + \theta_2 + \theta_3) = \frac{k}{2}[\cancel{\#}0]$$

Hence we get the solution

$$\theta_1 + \theta_2 + \theta_3 = 3ct^2 + 3D$$

We also $\theta_1 = \theta_2 = \theta_3 = ct^2 + D$

$$\therefore \quad \bar{E}(t) = \begin{bmatrix} ct^2 + D \\ C_2 \cos(\omega_1 t + \phi_1) \\ C_3 \cos(\omega_1 t + \phi_2) \end{bmatrix}$$

$$\omega_1 = \frac{\sqrt{3k}}{mr^2}$$

The general solution is

$$\bar{\theta}(t) = \bar{\bar{A}} \bar{E}(t).$$