

Rotation in an Asymmetric Multidimensional Periodic Potential due to Colored Noise

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(Received 28 May 1999)*

We analyze the motion of an overdamped classical particle in a multidimensional periodic potential, driven by an external noise. We demonstrate that in the steady state the presence of temporal correlations in the noise and spatial asymmetry within a period of the potential could lead to particle rotation. The rotation is a direct consequence of a change in the sign of the noise-induced drift motion in each dimension. By choosing different potentials, we can generate a variety of flow patterns from laminar drifts to rotations.

PACS numbers: 05.10.Gg, 05.40.-a, 87.10.+e

Recently there has been intense activity in the analysis of stochastically driven ratchets [1,2]. These ratchets are spatially periodic systems where a spatial asymmetry in the potential imposes a directionality, while “memory” effects from a temporally correlated stochastic force (colored noise) or explicit time dependences in the potential itself break detailed balance, thereby rectifying microscopic fluctuations to generate a unidirectional particle drift. This type of unidirectional drift has been verified through experiments on colloidal particles or polystyrene spheres [3,4], cold rubidium atoms in an asymmetric optical lattice [5], and quantum dots [6]. Predictions have also been made for SQUIDs [7]. The applications of these concepts have been manifold: this mechanism has been proposed as a possible explanation for the long-range cellular transport of motor proteins, utilizing temporal fluctuations at a microscopic scale to effectively transduce chemical energy into directed mechanical work [8]. In addition, such ideas are of interest in the nanoscale fabrication of devices.

Most recent analyses of ratchets have been limited to one-dimensional (or “linear”) molecular motors [8]. In this paper we show that for a particle in a multidimensional potential a time-correlated noise can break detailed balance and generate rotations. Beyond their intrinsic interest and novelty, such rotations could be used to design nonequilibrium molecular engines. To demonstrate rotations, we develop a Fokker-Planck equation in arbitrary dimensions for a weakly correlated Gaussian random noise and explicitly demonstrate the existence of rotation in the presence of spatial asymmetry and temporal correlations. Our argument can be simply described as follows: in the presence of spatial asymmetry in one dimension, a time-correlated noise is known to produce a drift [1,2]; the direction of the drift is determined by the sense of the potential asymmetry. For a potential in multiple dimensions, the sense of the potential asymmetry along one coordinate can be reversed by varying the other coordinates, leading to a change in sign (or reversal of asymmetry) of the potential. As indicated in Fig. 1, the coordinate-dependent reversal of the one-dimensional drifts could then conspire together to gener-

ate a rotation. Contrary to rotation generated in a potential, determined by the system’s initial conditions, the sense of our rotation is given *entirely* by the potential asymmetry and is *independent* of any initial conditions. Furthermore, removing either the potential asymmetry or the correlation in the noise destroys the rotation.

We put our general argument outlined above in a mathematical form as follows. We start with a Langevin equation describing an overdamped particle driven by a colored Gaussian noise of *arbitrary* correlation function in an *arbitrary* multidimensional periodic potential. For small correlation time, we derive a Fokker-Planck equation for the probability density of the particle. We then show that in steady state the curl of the current density cannot be zero everywhere in space as long as the correlation time is non-zero and the periodic potential is asymmetric. Finally we demonstrate that relaxing either of these two conditions leads to zero current density.

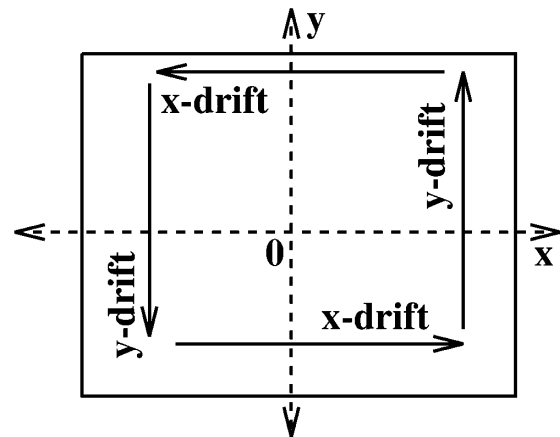


FIG. 1. Schematic description of rotation over one unit cell of a two-dimensional periodic potential, caused by spatial asymmetry and temporal correlations. One-dimensional drifts are produced by asymmetric potentials in x and y , in conjunction with correlated noise. The drifts switch directions owing to coordinate-dependent changes in overall sign of the potential asymmetry, and together produce rotation.

The Langevin equation describing the dynamics of an overdamped particle (damping coefficient γ set to unity) in a multidimensional space \vec{r} in the presence of a potential $U(\vec{r})$ and a noise $\vec{f}(t)$ is given by [9]

$$\dot{\vec{r}}(t) = \vec{W}(\vec{r}) + \vec{f}(t), \quad (1)$$

where the overdot represents a time derivative and $\vec{W} = -\vec{\nabla}U(\vec{r})$ is the force exerted by the potential $U(\vec{r})$. The noise is assumed to be Gaussian distributed, with a correlation time τ_i^c and strength D_i along the i th coordinate. Generalizing Fox's functional calculus approach [10], we express the probability distribution $P(\vec{r})$ in terms of a functional integral over different realizations of the noise. The noise in a particular direction i is temporally correlated through a correlation function C_i , while being uncorrelated with other directions $j \neq i$:

$$P(\vec{r}') = \int \mathcal{D}\vec{f} P[\vec{f}] \delta(\vec{r}' - \vec{r}(t)),$$

$$P[\vec{f}] = N \exp \left[- \int ds \int ds' \sum_{ij} \frac{f_i(s) f_j(s')}{2 \langle f_i(s) f_j(s') \rangle} \right], \quad (2)$$

$$\langle f_i \rangle = 0; \quad \langle f_i(t) f_j(t') \rangle = \frac{D_i}{\tau_i^c} C_i \left(\frac{|t - t'|}{\tau_i^c} \right) \delta_{ij}.$$

N is a normalization constant for P , and the subscripts i, j correspond to different spatial components of the multidimensional vectors.

The above representation of the probability distribution allows us to write down the equation of motion for the probability distribution $P(\vec{r})$ which implicitly depends on time through its coordinate and the Langevin equation (1). The equation takes the form of a continuity equation $\dot{P} = -\vec{\nabla} \cdot \vec{J}$. In the limit of weakly colored noise, the correlation time τ_i^c is much smaller than the diffusion time $\tau_i^D \equiv L_i^2/D_i$ over one period L_i of the potential, and the time $\tau_i^\gamma \equiv L_i^2/U_0$ for an overdamped particle to fall from a potential U_0 . Under these conditions, the current density \vec{J} satisfies the Fokker-Planck form [11]

$$J_i = W_i P - \frac{\partial}{\partial r_i} [\Theta_i P],$$

$$\Theta_i = D_i \left[1 + \mu_1^i \tau_i^c \mathbf{M}_{ii} - \frac{(\tau_i^c)^2}{2} \right. \\ \left. \times \mu_2^i (\mathbf{R} - \mathbf{M}^2)_{ii} + O(\tau_i^c)^3 \right], \quad (3)$$

where the matrix elements $\mathbf{M}_{ij} \equiv \partial W_i / \partial r_j$, $\mathbf{R}_{ij} \equiv \sum_k W_k \partial^2 W_i / \partial r_j \partial r_k$, and μ_1^i and μ_2^i are the first and second moments, respectively, of the correlation function C_i [11]. The role of color is thus to make the effective diffusion coefficient Θ_i position dependent in a well-defined manner [i.e., a function only of the potential $U(\vec{r})$].

We solve Eq. (3) for \vec{J} in steady state ($\dot{P} = 0$) and impose periodicity on P . In conjunction with the periodicity for U , this gives us a set of integral equations for \vec{J} . In

particular, in two dimensions, these read

$$\int_0^{L_x} dx J_x(x, y) e^{-\phi_x(x, y)} = P(0, y) \Theta_x(0, y) \\ \times [1 - e^{-\phi_x(L_x, y)}],$$

$$\int_0^{L_y} dy J_y(x, y) e^{-\phi_y(x, y)} = P(x, 0) \Theta_y(x, 0) \\ \times [1 - e^{-\phi_y(x, L_y)}], \quad (4)$$

$$\phi_x(x, y) = \int_0^x dz \frac{W_x(z, y)}{\Theta_x(z, y)},$$

$$\phi_y(x, y) = \int_0^y dz \frac{W_y(x, z)}{\Theta_y(x, z)},$$

where $L_{x,y}$ are the periods along x and y directions, respectively, and the coordinate zero is an arbitrary reference point on the x - y plane.

Simplifying Eq. (4) elucidates the role of the correlation and the asymmetry terms, which sit on the right-hand side. Expanding the square brackets to the right to the first significant order in τ_i^c ($\tau_1^c \equiv \tau_x^c, \tau_2^c \equiv \tau_y^c$) yields $[1 - e^{-\phi}] \approx \phi$, where ϕ is given by

$$\phi_x(L_x, y) = -\frac{(\tau_x^c)^2}{D_x} \int_0^{L_x} dx W_x(x, y) \\ \times \left[\frac{\mu_2^x}{2} \left(\frac{\partial W_x}{\partial y} \right)^2 - \left(\frac{3\mu_2^x}{4} - \frac{\mu_1^x}{2} \right) \right. \\ \left. \times W_x \frac{\partial^2 W_x}{\partial x^2} - \frac{\mu_2^x}{2} W_y \frac{\partial^2 W_x}{\partial x \partial y} \right] \quad (5)$$

and an analogous equation for $\phi_y(x, L_y)$. This term is nonvanishing as long as the noise is *correlated* ($\tau_i^c \neq 0$) and the potential in two dimensions is *asymmetric*, i.e., the integral over one period [Eq. (5)] is nonzero even though the integrand itself is periodic [12].

The origin of rotation can now be traced to the existence of color and asymmetry, which makes the integral in (5) and thus the right-hand side of Eq. (4) nonzero. This prevents \vec{J} from being identically zero or constant. Now, in steady state the divergence of \vec{J} is zero, so the only way for \vec{J} to not be identically zero within periodic boundary conditions is to have the curl of \vec{J} not identically zero. *This necessitates the current density to have local rotational fluxes.* On the other hand, if we take the limit of white noise ($\tau_i^c = 0, \forall i$) or make the integral in Eq. (5) vanish (symmetric potential), the right-hand side of Eq. (4) is zero, and then \vec{J} can have zero curl. In fact, we can make the statement stronger: for white noise, the effective diffusion constant $\Theta_i = D_i$ is position independent. In that case, the Fokker-Planck equation and the steady state lead to the following set of equations:

$$\vec{\nabla} \cdot \vec{J} = 0, \\ \vec{\nabla} \times (\vec{J} e^{U(\vec{r})/D}) = 0. \quad (6)$$

These equations, with periodic boundary conditions and detailed balance, lead to $\vec{J} \equiv 0$ [9] which corresponds to an equilibrium Maxwell-Boltzmann distribution for the probability distribution $P \sim \exp[-U/D]$. In other words, in white noise in steady state, there is no current density whatsoever. The damping in Eq. (1) breaks time-reversal symmetry in the system, while color breaks detailed balance, producing rotations.

Having thus established the necessary existence of rotational fluxes in a colored noise, we now move on to a concrete example for an asymmetric potential in two dimensions as shown in Fig. 2. The parameter a is the measure of the potential asymmetry. This potential is identical with respect to the transformation $x \leftrightarrow y$. We choose our correlation function for both the x and y variables to be decaying exponentials in time and assume in addition $\tau_i^c \equiv \tau^c$ and $D_i \equiv D$ to remove any superficial differences between the x and y directions.

We fix the boundary conditions of our potential by fixing $P(0, y) = P(x, 0) = \text{const}$ for the sake of definiteness. For a particular instantiation of our general arguments for nonzero rotation, we make an ansatz at this stage: we assume that J_x is a function of y only and J_y is a function of x . This ansatz is consistent with the form of Eq. (4) and trivially satisfies the steady-state condition $\vec{\nabla} \cdot \vec{J} = 0$. Then $J_x(y)$ is proportional to $\phi_x(L_x, y)$ of Eq. (5) divided by $\int_0^{L_x} dx \exp[-\phi_x(x, y)]$, with an analogous equation for $J_y(x)$. For the potential in Fig. 2, Eq. (5) simplifies to $\phi_x(L_x, y) = -3(\tau^c)^2 U_0^3 a \pi [4 \sin 2y + \sin 4y]/4D$.

Figure 3 shows the resulting field plot for the current density \vec{J} over one period of the potential. One immediately sees local rotational fluxes separated by saddle points. There are drifts along x for fixed y which change sign as described schematically in Fig. 1 and produce a local rotational flux. The role of *spatial asymmetry* and *temporal correlation* is clear by inspection of the expression for $\phi_x(L_x, y)$; the current density \vec{J} is zero if we set either τ^c

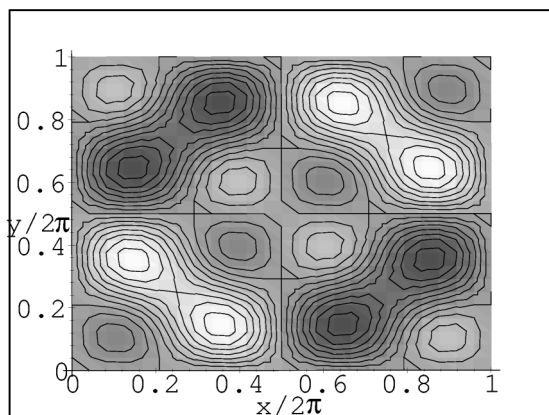


FIG. 2. Two-dimensional contour plot of a potential with translational asymmetry in x and y . The potential chosen is given by $U(x, y) = U_0[\sin x \sin y - a \sin 2x \sin 2y]$ with $U_0 = a = 1$. Dark areas correspond to valleys and bright patches correspond to hills in the potential landscape.

or a to be zero. In other words, the correlated noise breaks detailed balance, thereby exploiting the spatial asymmetries in the potential to produce local drifts and rotations.

The relation between the potential profile in Fig. 2 and the flow pattern in Fig. 3 can be summarized as follows: for a fixed x (y) coordinate, the sense of the drift along y (x) is given by the asymmetry integral in Eq. (5). Note that the rotations take the particle in and out of potential hills and valleys, implying that energy is not conserved in the rotation process. The particle does not follow any of the equipotentials; neither does the direction of the rotation depend on the initial conditions of the system. The noise-driven rotation is a nonequilibrium process which disobeys the fluctuation-dissipation theorem, and the sense of the rotation is determined entirely by the moments of the correlation function and potential asymmetries.

Varying the potential leads to a whole range of steady-state flow patterns, including laminar flow (global drift), global rotation (over one unit cell of the potential), and their various combinations. To get a drift in any one coordinate, for example, we need an asymmetric potential (such as a ratchet) in that direction. Combining asymmetric potentials in various ways, we can generate the following classes of noise-driven fluxes [11]:

Flow pattern	Potential form
Rotation	Ratchets coupled in x and y
Laminar flow	Decoupled ratchets in x and y
Rotation + net drift	Coupled ratchets asymmetric under $x \leftrightarrow -x, y \leftrightarrow -y$

Our arguments are generalizable to higher dimensions; breaking other kinds of symmetries in N dimensions will lead to a complex network of flow patterns.

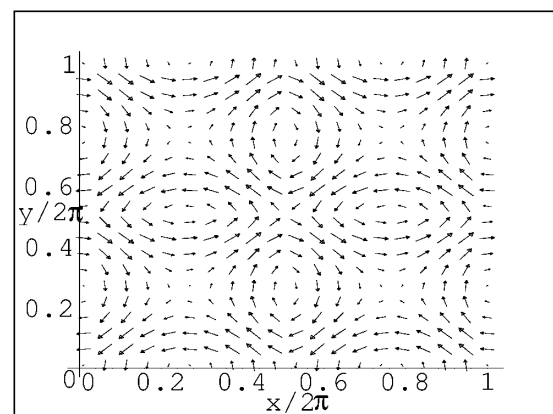


FIG. 3. Local rotations (\vec{J}) produced by drifts arising out of translational asymmetries in the potential in Fig. 2 and correlation in the noise. There is no global rotation, and the steady-state flow consists only of cycles and saddle points. At fixed values of x there are global drifts in y that reverse sign so as to generate the rotations. The arrow lengths are proportional to $(\tau^c)^2 U_0^3 a/D$, so that as a or $\tau^c \rightarrow 0$ the arrow length shrinks to zero and the local current densities vanish.

The broken detailed balance and spatial asymmetry essential for current generation could be introduced in a variety of ways; we have considered correlated noise and asymmetry in the potential. One could obtain similar results by introducing both spatial asymmetry and temporal correlation in the potential, leading to currents in a “flashing potential” [2]. On the other hand, one could include both of these in the noise itself and leave the potential symmetric with respect to x and y individually. This can be accomplished, for example, by a noise that has more kicks on average in one direction than the other, thereby incorporating both broken spatial symmetry and detailed balance [13]. In essence, the noise has to be out of equilibrium with the system to produce current [14].

One-dimensional ratchets have been explored under a variety of experimental conditions [3–5]. To observe noise-induced rotation in higher dimensions, we now propose a realistic experimental setup. Consider a system of $0.07\text{--}0.1\ \mu\text{m}$ charged, fluorescent polystyrene beads suspended in an aqueous solution at room temperature in a two-dimensional potential. The periods of the potential are chosen to be $L_x = L_y \approx 1\ \mu\text{m}$, with an asymmetry fraction ~ 0.4 and maximum energy $U_0 \approx 75\ \text{meV}$. A set of crisscrossing electrodes is lithographically patterned to generate the two-dimensional ratchet potential similar to the one-dimensional potential in Ref. [3]. Alternatively we can build a ratchet optically as in Ref. [4], for example, by passing a low intensity laser light through a patterned reticle. For a colored noise with a $10\text{--}40\ \text{Hz}$ bandwidth generated electronically or optically, the particles will settle into slow circular orbits of $\sim 1\ \mu\text{m}$ diam. Our calculations give us an estimated period of rotation of $1\text{--}10\ \text{h}$ [11]. Such fluorescent vortex patterns should be observable using a microscope.

We have analyzed the steady-state dynamics of an overdamped classical particle in an arbitrary multidimensional potential driven by a noise with an arbitrary correlation function. For nonzero temporal correlations and asymmetries in the potential, current production in terms of rotations and drifts is expected. We have demonstrated rotation explicitly in two dimensions in the limit of small correlation times. Such rotations could be prototypes of periodic nonequilibrium processes such as molecular engines. By suitably tailoring the potential, one can generate a host of nonequilibrium flow patterns of the particle, including global drifts and rotations in combination. We have proposed a way of monitoring rotations in a realistic experimental setup.

We thank O. Pierre-Louis for suggesting the problem and C. Jayaprakash and F. Jülicher for useful discussions. A.W.G. thanks The Ohio State University Presidential Fellowship and DOE for support. S. V. K. thanks the Department of Energy, Basic Energy Sciences, Division of Materials Sciences, through The Ohio State University and also Grant No. DEFG02-96ER45439 through the University of Illinois at Urbana-Champaign, Seitz Materials Research Laboratory, for support.

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