Chapter 15: Oscillations

Oscillations are motions that repeat themselves.

springs, pendulum, planets, molecular vibrations/rotations

Frequency f : number of oscillations in 1 sec.

unit: 1 hertz (Hz) = 1 oscillation per second = 1 s^{-1}

Period T : the time for one complete oscillation.
It is the inverse of frequency.

$$T = 1/f \qquad \text{or} \qquad f = 1/T$$

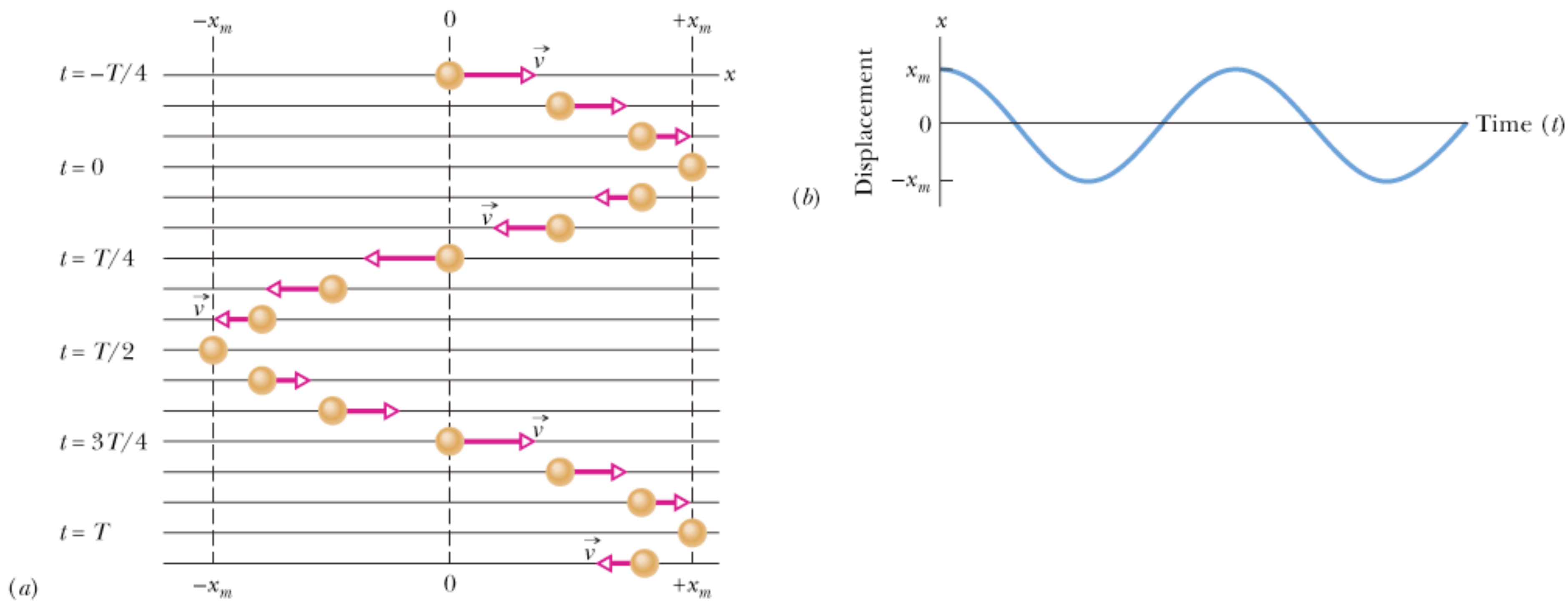


FIGURE 15-1 *(a)* A sequence of "snapshots" (taken at equal time intervals) showing the position of a particle as it oscillates back and forth about the origin of an x axis, between the limits $+x_m$ and $-x_m$. The vector arrows are scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at $\pm x_m$. If the time t is chosen to be zero when the particle is at $+x_m$, then the particle returns to $+x_m$ at $t = T$, where T is the period of the motion. The motion is then repeated. *(b)* A graph of x as a function of time for the motion of *(a)*.

Simple Harmonic Motion

Simple harmonic motion (SHM) is the oscillation in which the displacement x(t) is in the form of

$$x(t) = x_m\cos(\omega t + \varphi)$$

x_m, ω and φ are constants

x_m: amplitude or maximum displacement, $x_{max} = A$

$\omega t + \varphi$: phase; φ: phase constant or phase angle

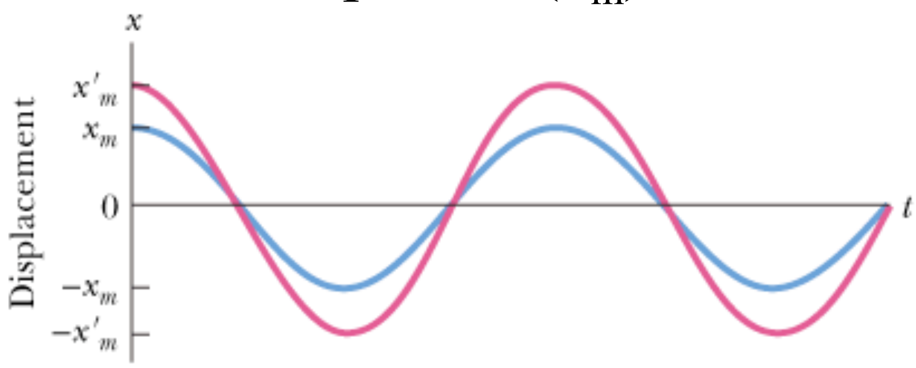
What is ω? since $x(t) = x(t + T)$ (at same place after 1 period)

so $x_m\cos(\omega t) = x_m\cos(\omega(t+T))$ (let $\varphi = 0$)

thus $\omega(t+T) = \omega t \rightarrow \omega T = 2\pi$

$\omega = 2\pi/T = 2\pi f$ ω is the angular frequency (unit: rad/s)

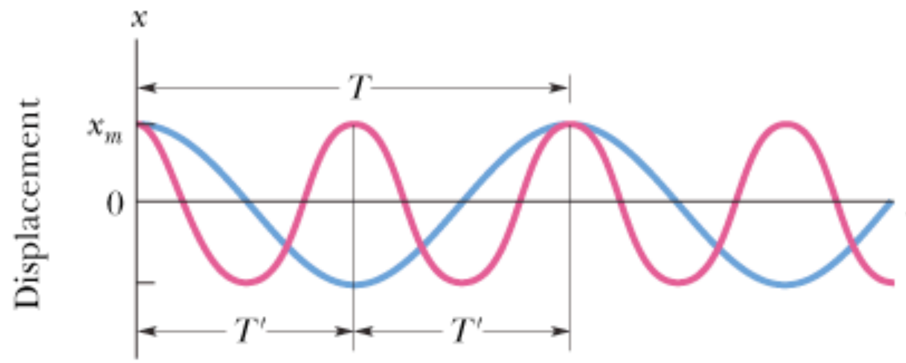
Different Amplitude (x_m)



(a)

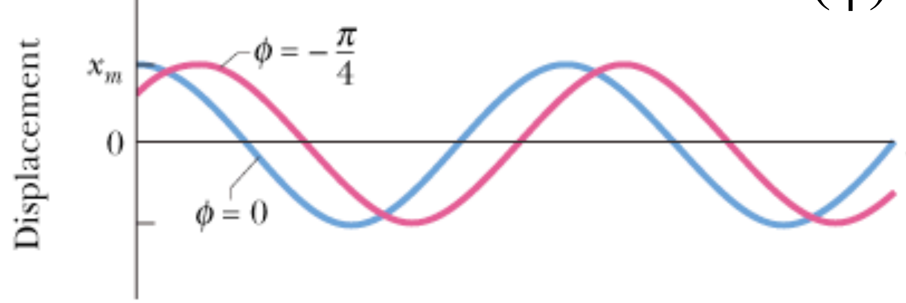
$$\overbrace{x(t)}^{\text{Displacemnt at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos\big(\overbrace{\underbrace{\omega}_{\text{Angular frequency}}\, \underbrace{t}_{\text{Time}} + \underbrace{\phi}_{\text{Phase constant or phase angle}}}^{\text{Phase}}\big)$$

Different $\omega = 2\pi/T$



(b)

Different Phase (φ)



(c)

- Displacement of SHM:

  $x(t) = x_m \cos(\omega t + \varphi)$

- Velocity of SHM:

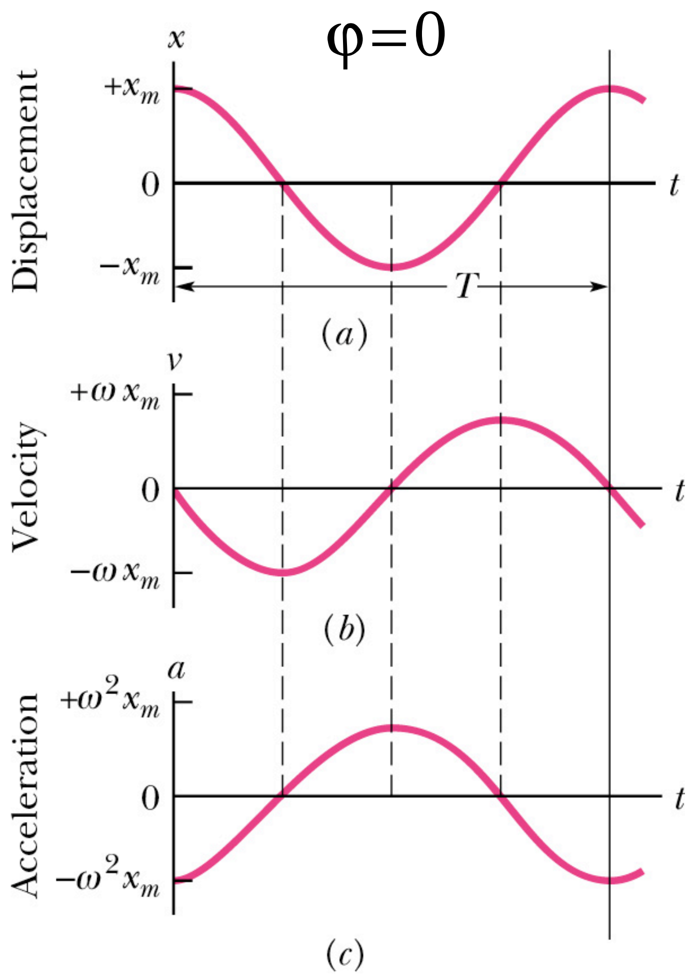
  $v(t) = dx(t)/dt = -\omega x_m \sin(\omega t + \varphi)$

  velocity amplitude: $v_{max} = \omega x_m$

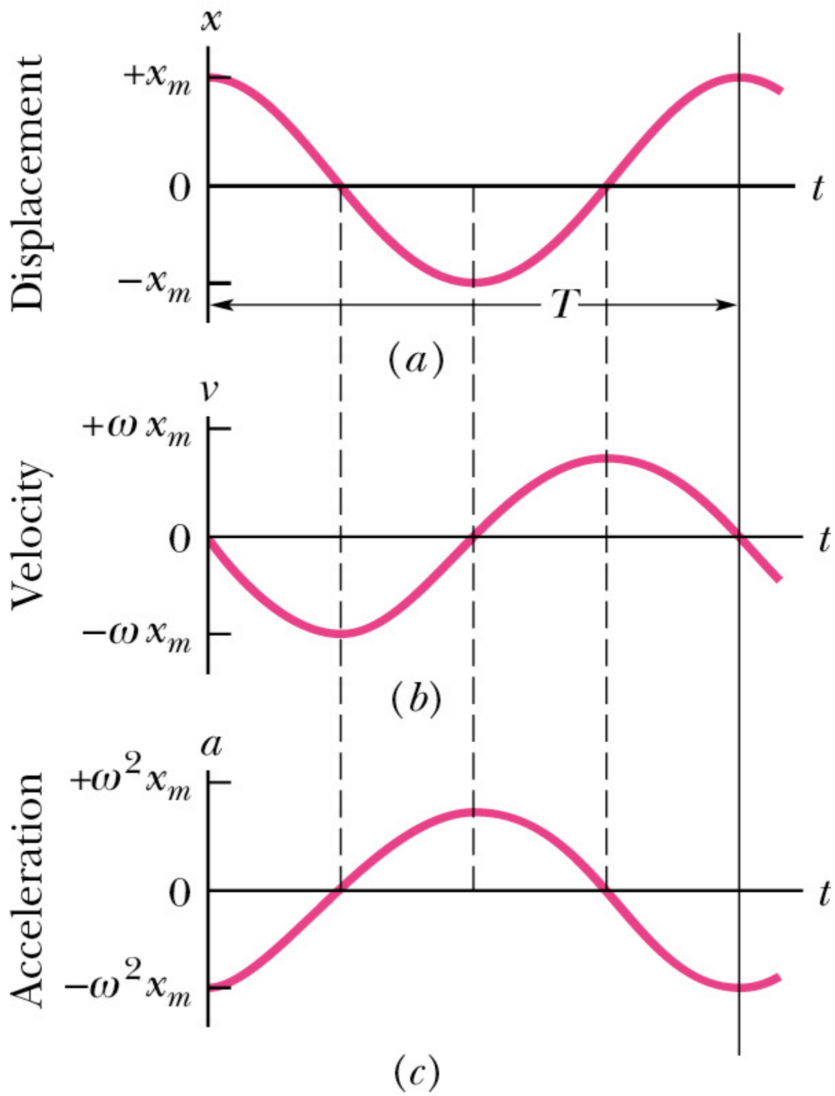
- Acceleration of SHM:

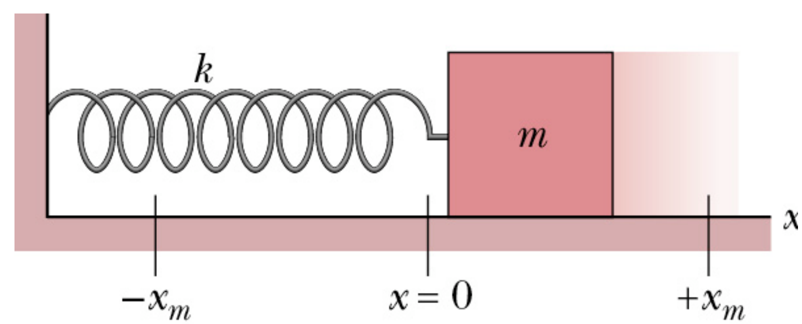
  $a(t) = dv(t)/dt = -\omega^2 x_m \cos(\omega t + \varphi)$

  $a(t) = -\omega^2 x(t)$: $a_{max} = \omega^2 x_m$



In SHM, the acceleration is proportional to the displacement but opposite in sign, and the two quantities are related by the square of the angular frequency.


Displacement
Velocity
Acceleration
x
$+x_m$
0
$-x_m$
t
T
(a)
v
$+\omega x_m$
$-\omega x_m$
(b)
a
$+\omega^2 x_m$
$-\omega^2 x_m$
(c)


k
m
x
$-x_m$
$x = 0$
$+x_m$

- The force law for SHM:

  since $a(t) = -\omega^2 x(t)$, $F = ma = -m\omega^2 x = -\underline{(m\omega^2)}x$

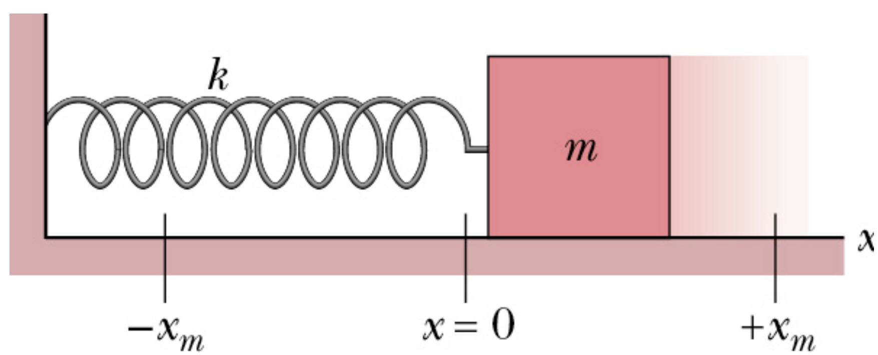
  spring force fits this criteria: $F = -kx$

- Therefore, the block-spring system is a linear simple harmonic oscillator with

$$k = m\omega^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$



- Energy in SHM

Potential energy:

$U(t) = \frac{1}{2} kx^2 = \frac{1}{2} kx_m^2 \cos^2(\omega t + \varphi)$

Kinetic energy:

$K(t) = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \varphi)$

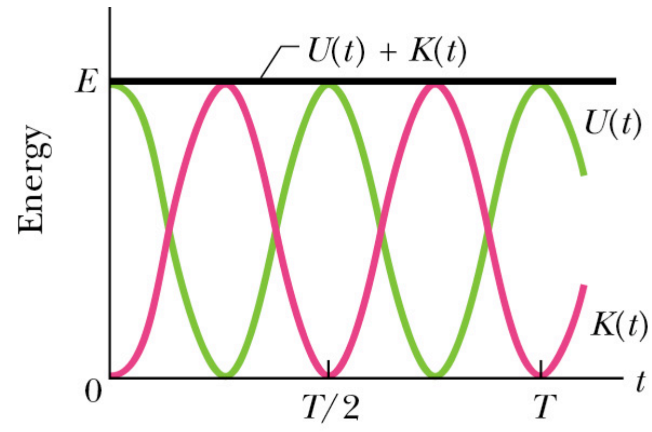
since $k/m = \omega^2$

$K(t) = \frac{1}{2} kx_m^2 \sin^2(\omega t + \varphi)$

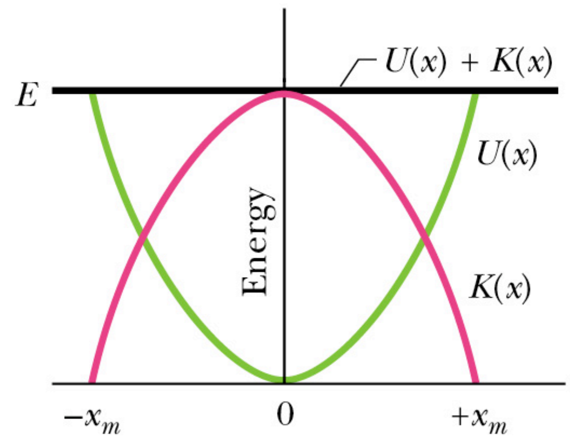
Mechanical energy:

$E = U + K = \frac{1}{2} kx_m^2[\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)] = \frac{1}{2} kx_m^2$

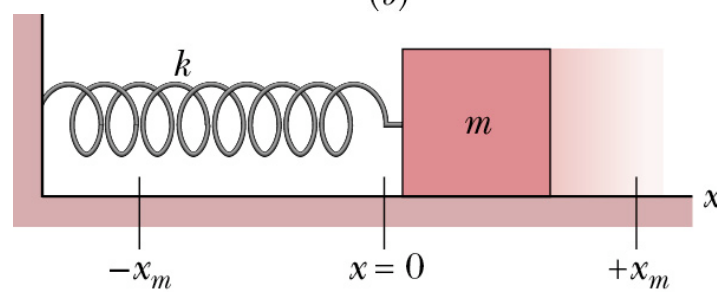
Mechanical energy is indeed a constant, and is independent of time .



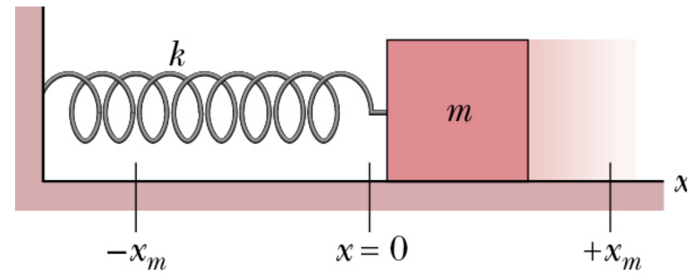
(a)



(b)



The block has a kinetic energy of 3.0J and the spring has an elastic potential energy of 2.0J when the block is at x = +2.0 m.



A) What is the kinetic energy when the block is at x = 0?

$$E = \underbrace{U(x,t)}_{2.0} + \underbrace{K(x,t)}_{3.0} = 5.0J = \tfrac{1}{2}\, mv^2 + \underbrace{U(0,t)}_{0}$$

B) What is the potential energy when the block is at x = 0?

$$U(x,t) = \tfrac{1}{2}\, kx^2 = \tfrac{1}{2}\, kx_m^{\,2} \cos^2(\omega t + \varphi) = 0$$

C) What is the potential energy when the block is at x = −2.0m?

$$U(x,t) = U(-x,t') = 2.0J$$

D) What is the potential energy when the block is at x = $-x_m$?

$$E = U(t) + \underbrace{K(t)}_{0} = \tfrac{1}{2}\, kx_m^{\,2}[\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)] = \tfrac{1}{2}\, kx_m^{\,2} = 5.0J$$

An angular simple harmonic oscillator

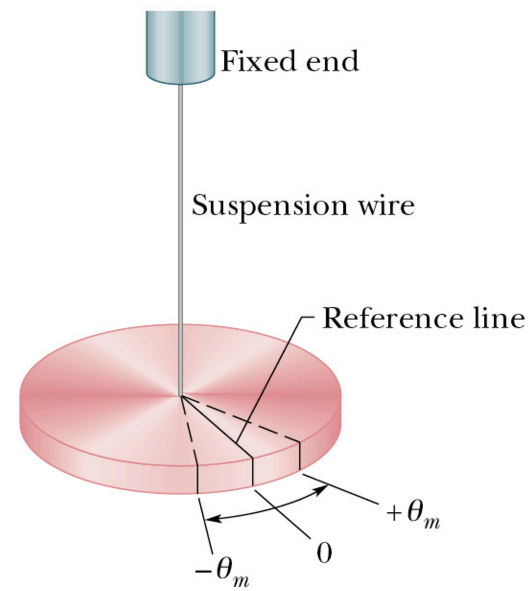
- Torsion (twisting) pendulum

  with restoring torque:

  $\tau = -\kappa\theta$ (angular form of Hooke's Law)

  compare to $F = -kx$ where

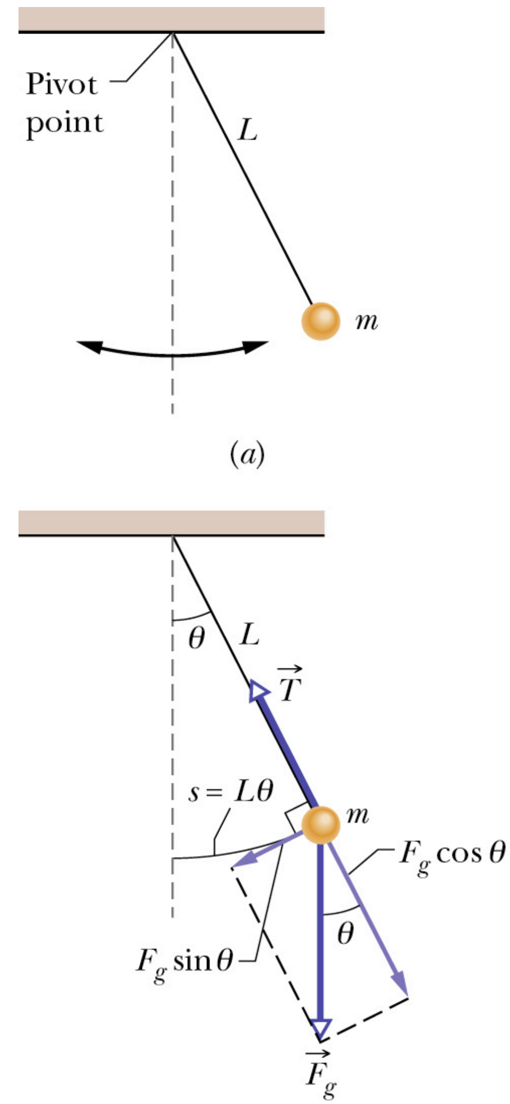
  $$T = 2\pi\sqrt{\frac{m}{k}}$$ since $k = m\omega^2$, $\omega = 2\pi/T$



Thus, for angular SHM we have:

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$ (subsituting, by analogy, I for m, and κ for k)

Another SHM Device



The simple pendulum

restoring torque: $\tau = -L(mg\sin\theta)$

"–" indicates that τ acts to reduce θ.

For small θ, $\sin\theta \sim \theta$ thus: $\tau = -Lmg\theta$

Remembering $\tau = I\alpha$, we have:

$$\alpha = -(Lmg/I)\theta,$$

hallmark of angular SHM (compare to: $a = -\omega^2 x$)

Thus, $\omega = \sqrt{Lmg/I}$

I of "bob" about swinging point = mL^2, and $\omega = 2\pi/T$

So, $g = (2\pi/T)^2 L$ *–can be used to measure g!*

Alternatively…

The simple pendulum

restoring torque: $\tau = -Lmg\theta$

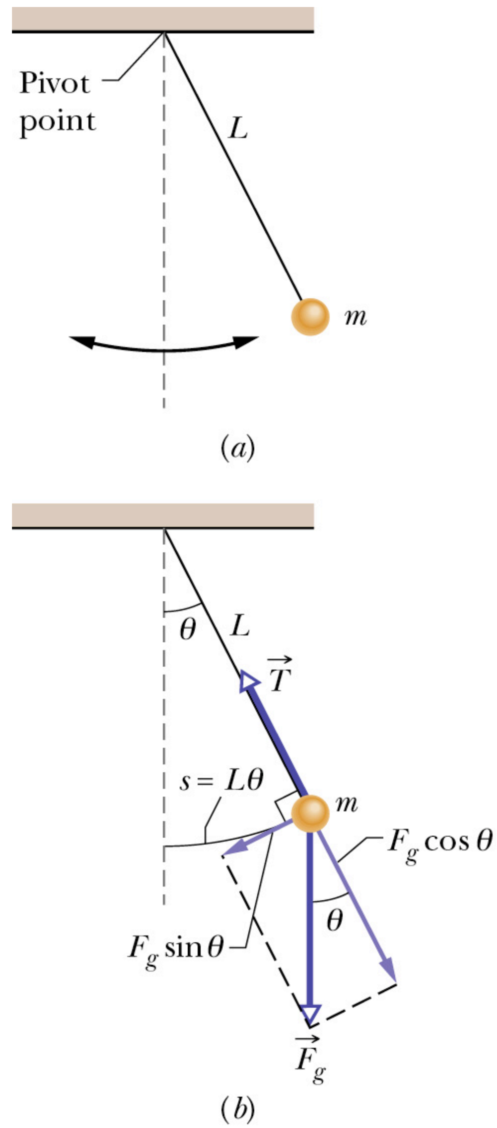
and $\tau = -\kappa\theta$; (angular form of Hooke's Law)

so: $\kappa = mgL$,

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

This can be used to measure g

$g = (2\pi/T)^2L$ (same result)



- **Another SHM Device - The physical pendulum**

(real pendulum with arbitrary shape)

h: distance from pivot point O to the center of mass.

Restoring torque:

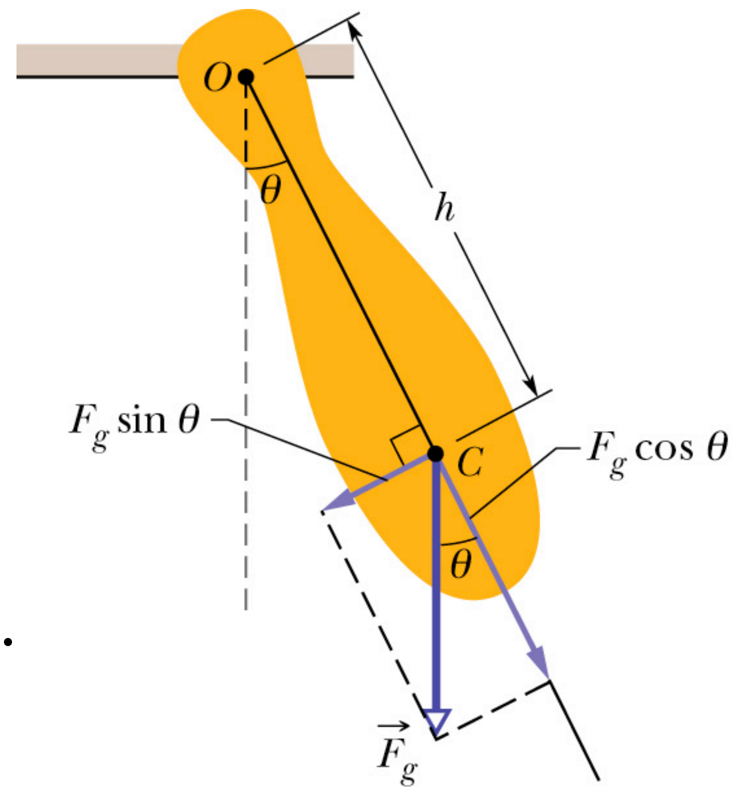
$\tau = -h\,(mg\sin\theta) \sim (mgh)\theta$ (if θ is small)

$\kappa = mgh$ (since $\tau = -\kappa\theta$)

therefore:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{I}{mgh}}$$

I: rotational inertia with respect to the rotation axis thru the pivot.



Sample Problem

Which of the following relationships between the force F on a particle and the particle's position x implies simple harmonic oscillation?

1. $F = -5x$
2. $F = -400x^2$
3. $F = 10x$
4. $F = 3x^2$
5. none of the above

Daily Quiz, March 20, 2006

$$F = ma(t) = m\,dv(t)/dt = -m\omega^2 x_m \cos(\omega t + \varphi) = -(m\omega^2)x(t)$$

$$m\omega^2 = 5.0$$

Which of the following relationships between the force F on a particle and the particle's position x implies simple harmonic oscillation?

1) $F = -5x$
2) $F = -400x^2$
3) $F = 10x$
4) $F = 3x^2$
5) none of the above

Linear SHM Summary

$F(t) = -kx$ Linear SHM

$F(t) = -m\omega^2 x(t)$ where $x(t) = x_m \cos(\omega t + \varphi)$

So, $a(t) = -\omega^2 x(t)$ "Hallmark" of SHM

With that info:

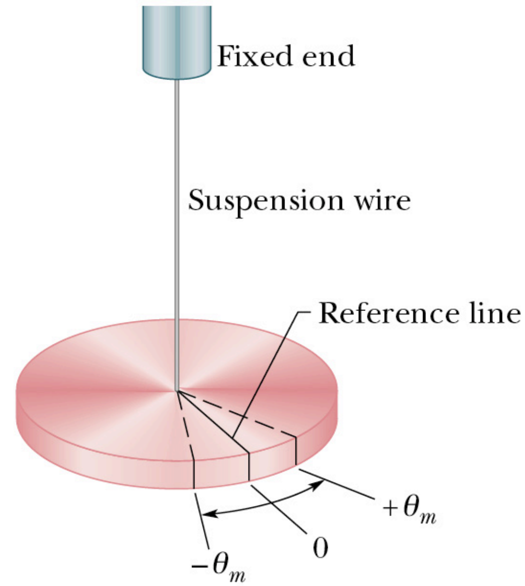
$m\omega^2 = k \rightarrow \omega = \sqrt{(k/m)}$

And since $\omega = 2\pi f$, and $f = 1/T \rightarrow T = 2\pi/\omega$

so $T = 2\pi\sqrt{(m/k)}$ (units of seconds)

The period T is the easiest parameter to observe experimentally

Torsion Pendulum



$\tau = -\kappa\theta$ Angular SHM (ASHM)

Compare to

$F(t) = -kx$ Linear SHM (LSHM)

$\tau = -\kappa\theta$ is the angular form of Hooke's law

Since $T = 2\pi\sqrt{(m/k)}$ (linear)

$T = 2\pi\sqrt{(I/\kappa)}$ (angular – I is analog to mass, and κ is like k. κ is a property of the wire, while k is a property of a spring)

We still have $\omega=2\pi f$, and $f = 1/T \rightarrow T = 2\pi/\omega$

(to develop other relations….)

Figure 15-8*a* shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped objec which we call object X, is then hung from the same wire, as in Fig. 15-8*b*, and its period T_b is found to be 4.7 s. What is the rotational inertia of object X about its suspension axis?

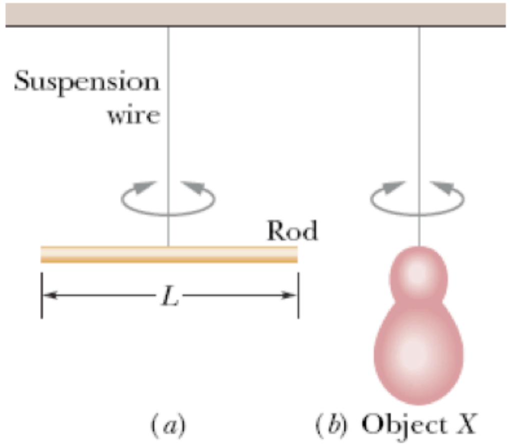


FIGURE 15-8 Two torsion pendulums, consisting of (*a*) a wire and a rod and (*b*) the same wire and an irregularly shaped object.

KEY IDEA

The rotational inertia of either the rod or object X is related to the measured period by Eq. 15-23.

Calculations: In Table 10-2e, the rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as $\frac{1}{12}mL^2$. Thus, we have, for the rod in Fig. 15-8*a*,

$$\begin{aligned} I_a &= \frac{1}{12}mL^2 = \left(\frac{1}{12}\right)(0.135\,\text{kg})\,(0.124\,\text{m})^2 \\ &= 1.73\times10^{-4}\,\text{kg}\cdot\text{m}^2. \end{aligned}$$

Now let us write Eq. 15-23 twice, once for the rod and once for object X:

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

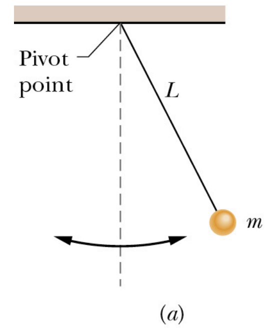
The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b. The result is

$$\begin{aligned} I_b &= I_a\frac{T_b^2}{T_a^2} = \left(1.73\times10^{-4}\,\text{kg}\cdot\text{m}^2\right)\frac{(4.76\,\text{s})^2}{(2.53\,\text{s})^2} \\ &= 6.12\times10^{-4}\,\text{kg}\cdot\text{m}^2. \end{aligned} \qquad \text{(Answer)}$$

Simple Pendulum (we can go a bit farther)

$\tau = -\kappa\theta$ Angular SHM

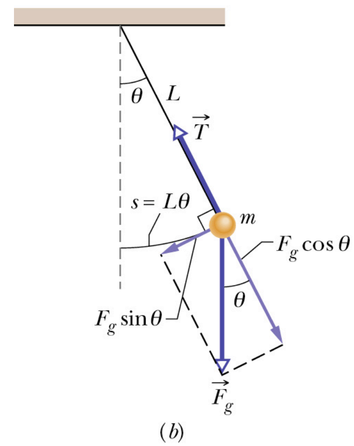


We can analyze the motion more thoroughly using:

$\tau = I\alpha$ ($T = 2\pi\sqrt{(I/\kappa)}$ is still "good")

$\tau = -Lmg\theta$,

(since $\tau = r \times F$, and $\sin(\theta) = \theta$ for small θ)



So, $I\,\alpha = -Lmg\theta$, or: $\alpha(t) = -(Lmg/I)*\theta(t)$

Which can be compared to

$a = -\omega^2 x$ (hallmark of LSHM)

So, for ASHM, $\omega = \sqrt{(Lmg/I)}$, or $\omega = \sqrt{(g/L)}$ since $I = mL^2$

We still have $\omega=2\pi f$, and $f = 1/T \rightarrow T = 2\pi/\omega$, so $T = 2\pi\sqrt{(L/g)}$

Sample Problem 15-5

In Fig. 15-11*a*, a meter stick swings about a pivot point at one end, at distance *h* from the stick's center of mass.

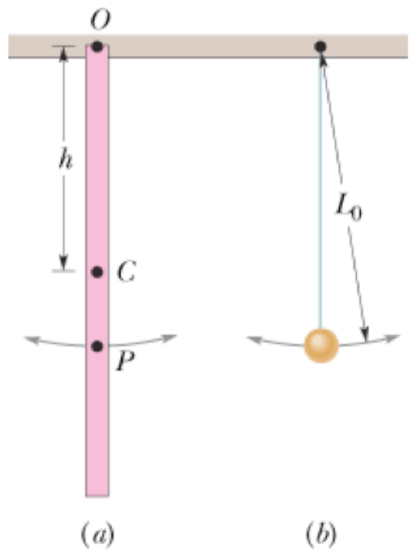


FIGURE 15-11 (*a*) A meter stick suspended from one end as a physical pendulum. (*b*) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. Point *P* on the pendulum of (*a*) marks the center of oscillation.

(a) What is the period of oscillation *T*?

KEY IDEA

The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

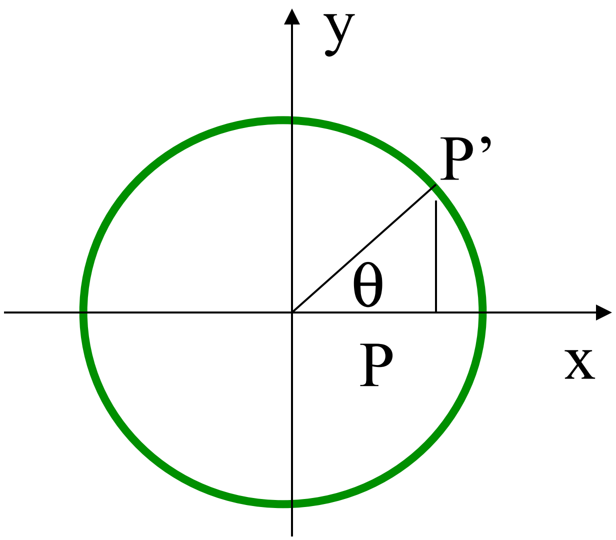
Calculations: The period for a physical pendulum is given by Eq. 15-29, for which we need the rotational inertia *I* of the stick about the pivot point. We can treat the stick as a uniform rod of length *L* and mass *m*. Then Eq. 15-30 tells us that $I = \frac{1}{3}mL^2$, and the distance *h* in Eq. 15-29 is $\frac{1}{2}L$. Substituting these quantities into Eq. 15-29, we find

$$\begin{aligned} T &= 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg\left(\frac{1}{2}L\right)}} = 2\pi\sqrt{\frac{2L}{3g}} \qquad (15\text{-}32) \\ &= 2\pi\sqrt{\frac{(2)(1.00\,\mathrm{m})}{(3)\left(9.8\,\mathrm{m/s^2}\right)}} = 1.64\,\mathrm{s}. \qquad \text{(Answer)} \end{aligned}$$

Note the result is independent of the pendulum's mass *m*.

Simple harmonic motion and circular motion

- Circular motion of point P':
  angular velocity: ω
  $\theta = \omega t + \varphi$
  P is the projection of P'on x-axis:
  $x(t) = x_m \cos(\omega t + \varphi)$ SHM



- P': uniform circular motion
  P: simple harmonic motion

Read section 15-7 in the book

Figure 15-14*a* gives an example. It shows a *reference particle P′* moving in uniform circular motion with (constant) angular speed ω in a *reference circle*. The radius x_m of the circle is the magnitude of the particle's position vector. At any time *t*, the angular position of the particle is $\omega t + \phi$, where φ is its angular position at $t = 0$.

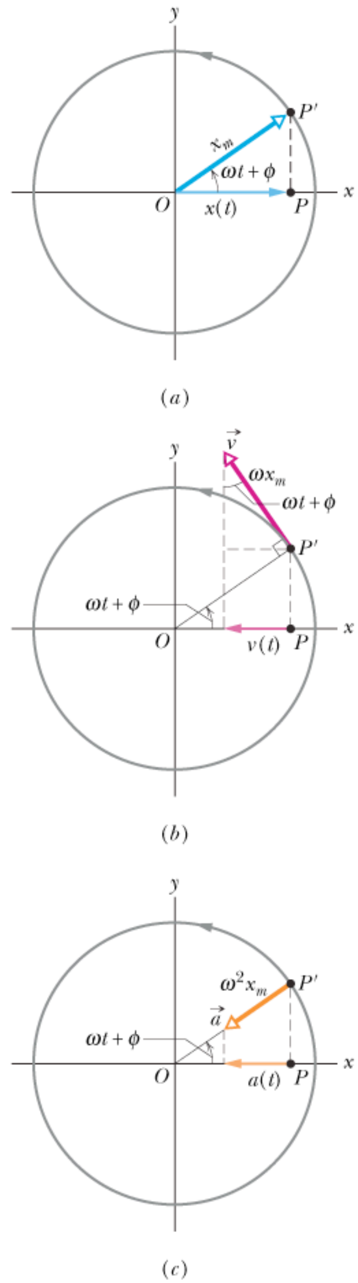
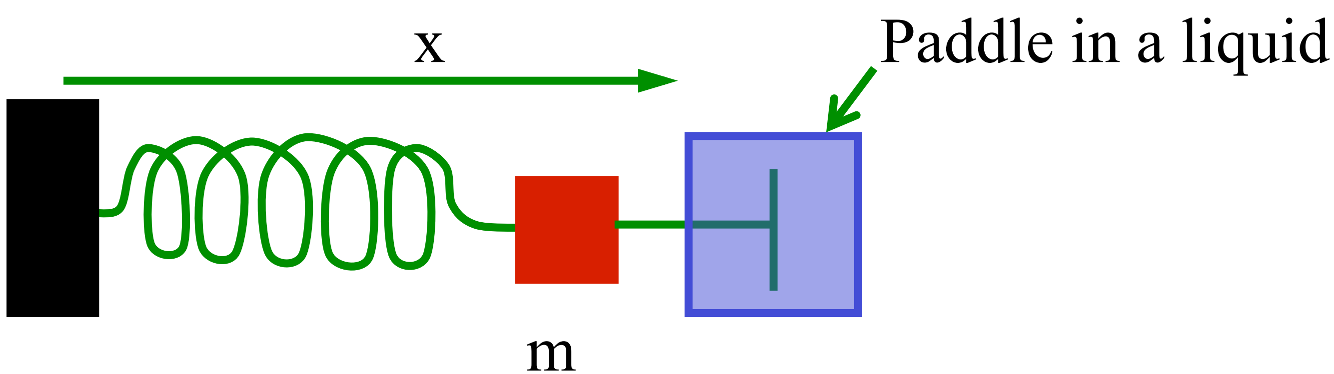


FIGURE 15-14 (*a*) A reference particle *P′* moving with uniform circular motion in a reference circle of radius x_m. Its projection *P* on the *x* axis executes simple harmonic motion. (*b*) The projection of the velocity $\vec{v}$ of the reference particle is the velocity of SHM. (c) The projection of the radial acceleration $\vec{a}$ of the reference particle is the acceleration of SHM.

Damped simple harmonic motion

- When the motion of an oscillator is reduced by an external force, the oscillator or its motion is said to be damped.
- The amplitude and the mechanical energy of the damped motion will decrease exponentially with time.



Damped simple harmonic motion

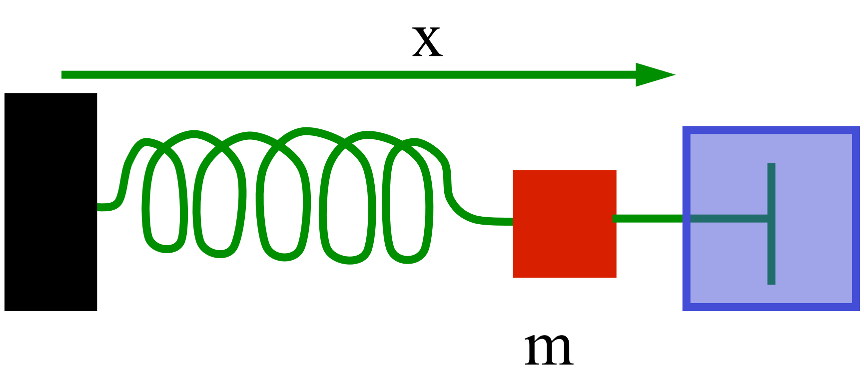
Proportional to velocity, but in the opposite direction

Spring Force $$F_s(x,t) = -kx$$

Damping Force $$F_d(x,t) = -b\,v(t) = -b\left(\frac{dx}{dt}\right)$$

Response Function (Newton's Second Law)

$$\sum_i F_i(x,t) = ma = m\left(\frac{d^2x}{dt^2}\right) \Rightarrow -kx - b\left(\frac{dx}{dt}\right)$$



Damped simple harmonic motion

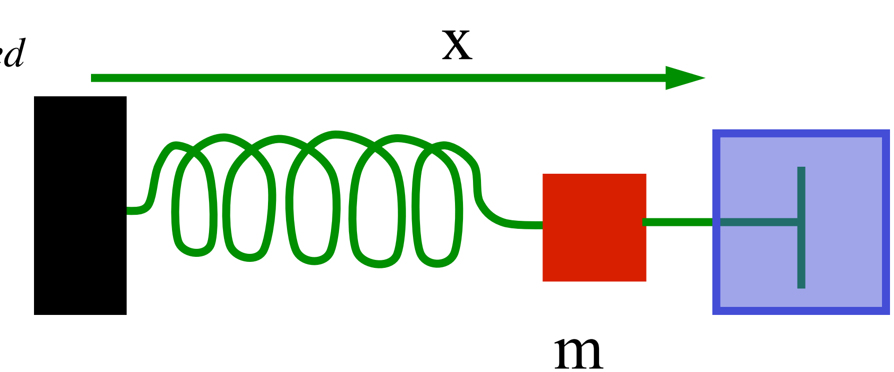
Response Function (Newton's Second Law)

Rearranging: $$m\left(\frac{d^2x}{dt^2}\right)+b\left(\frac{dx}{dt}\right)+kx=0$$

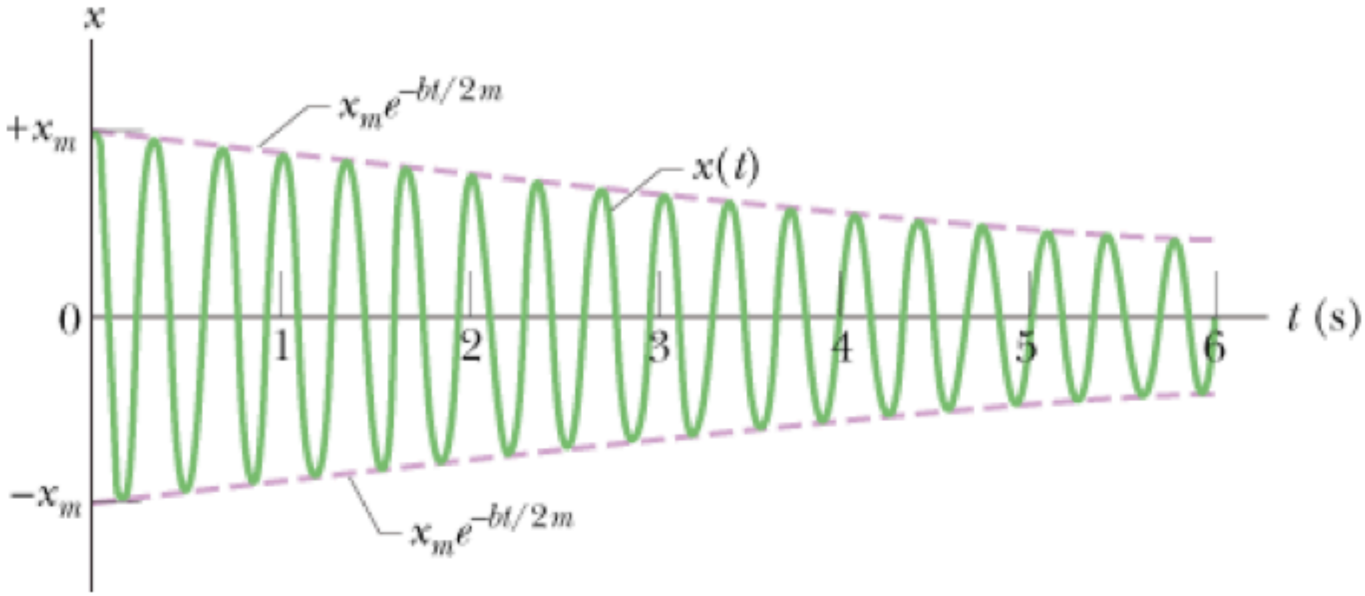
This differential equation has the solution:

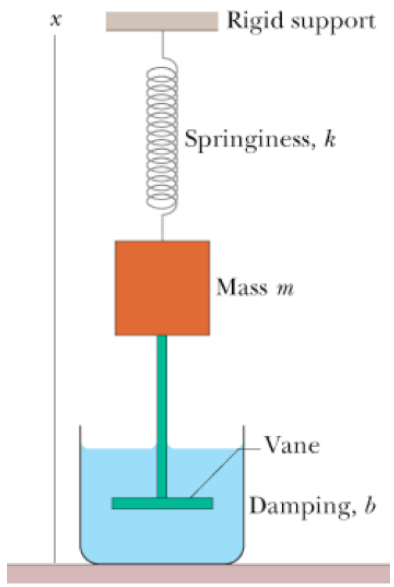
$$x(t)_{damped}=x_m e^{-bt/2m}\cos(\omega' t+\varphi)$$ where, $$\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

$$x(t)_{damped}=e^{-bt/2m}x(t)_{undamped}$$



Damped simple harmonic motion





Total Energy for LSHM = U + K = $\frac{1}{2} k x_m^2$
Energy for DLSHM:

$E_{tot} \approx \frac{1}{2} k x_m^2 e^{-bt/m}$, for small damping

Forced Oscillation and Resonance

- Free oscillation and forced oscillation
- For a simple pendulum, the natural frequency $\omega_0 = \sqrt{\frac{g}{l}}$
- Now, apply an external force: $F = F_m \cos(\omega_d t)$

  ω_d, driving frequency

  x_m depend on ω_0 and ω_d,

  when $\omega_d = \omega_0$, x_m is about the largest this is called **resonance**.

  examples : push a child on a swing, air craft design, earthquake

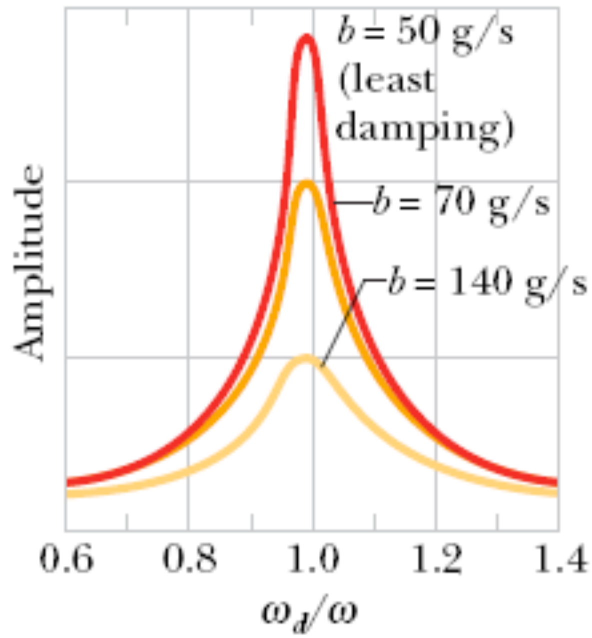


FIGURE 15-17 The displacement amplitude x_m of a forced oscillator varies as the angular frequency ω_d of the driving force is varied. The curves here correspond to three values of the damping constant b.