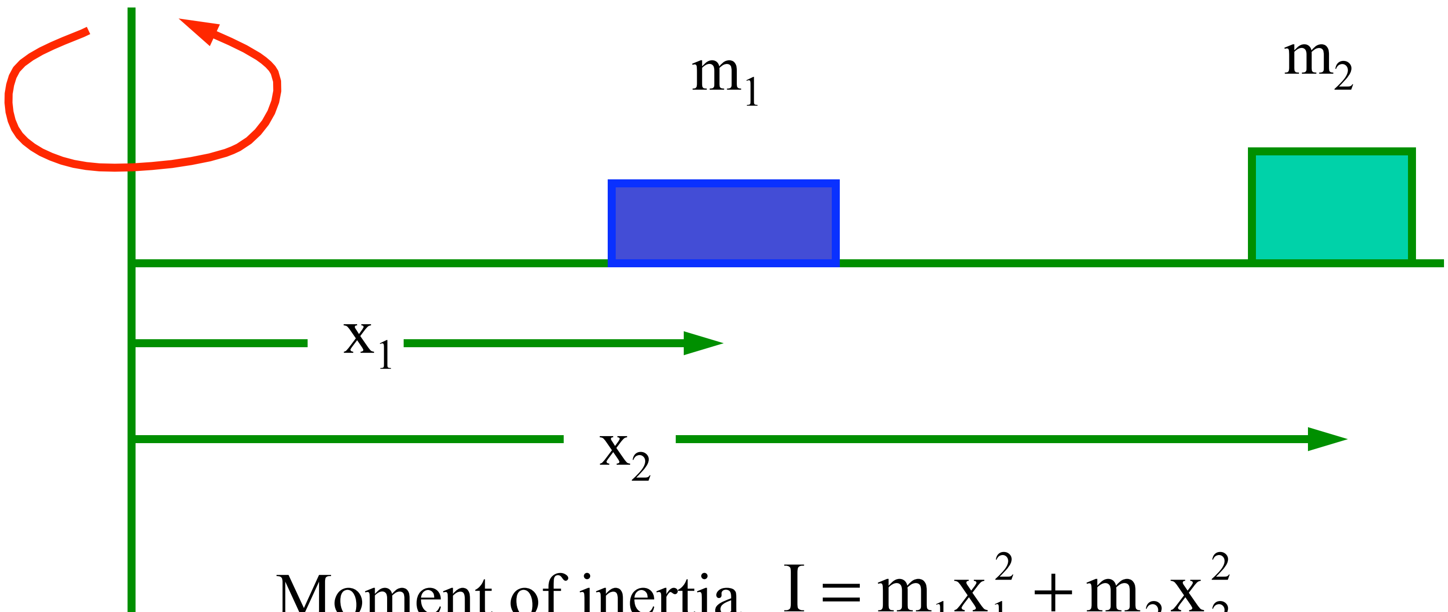
Rotational inertia (moment of inertia)

Define **rotational inertia** (**moment of inertia**) to be

$$I = \Sigma m_i r_i^2 \quad \text{or} \quad I = \int r^2 dm$$

r_i : the perpendicular distance between m_i and the given rotation axis



Moment of inertia $I = m_1 x_1^2 + m_2 x_2^2$

Rotational inertia (moment of inertia)
Hoop rotating about a central axis

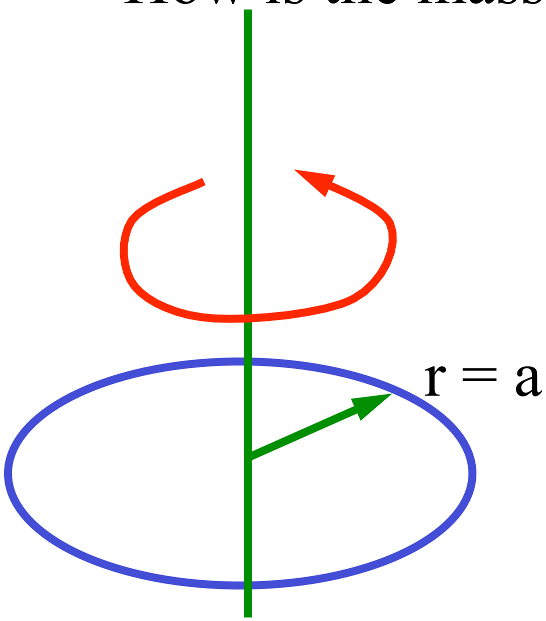
Define **rotational inertia** (**moment of inertia**) to be

$$I = \int r^2 dm$$

r_i : the perpendicular dist. between m_i and the rotation axis

How is the mass distributed on the hoop? >>>> $dm/M = rd\theta/2\pi r$

or $dm = \rho\, r\, d\theta$, where $\rho = M/2\pi r$



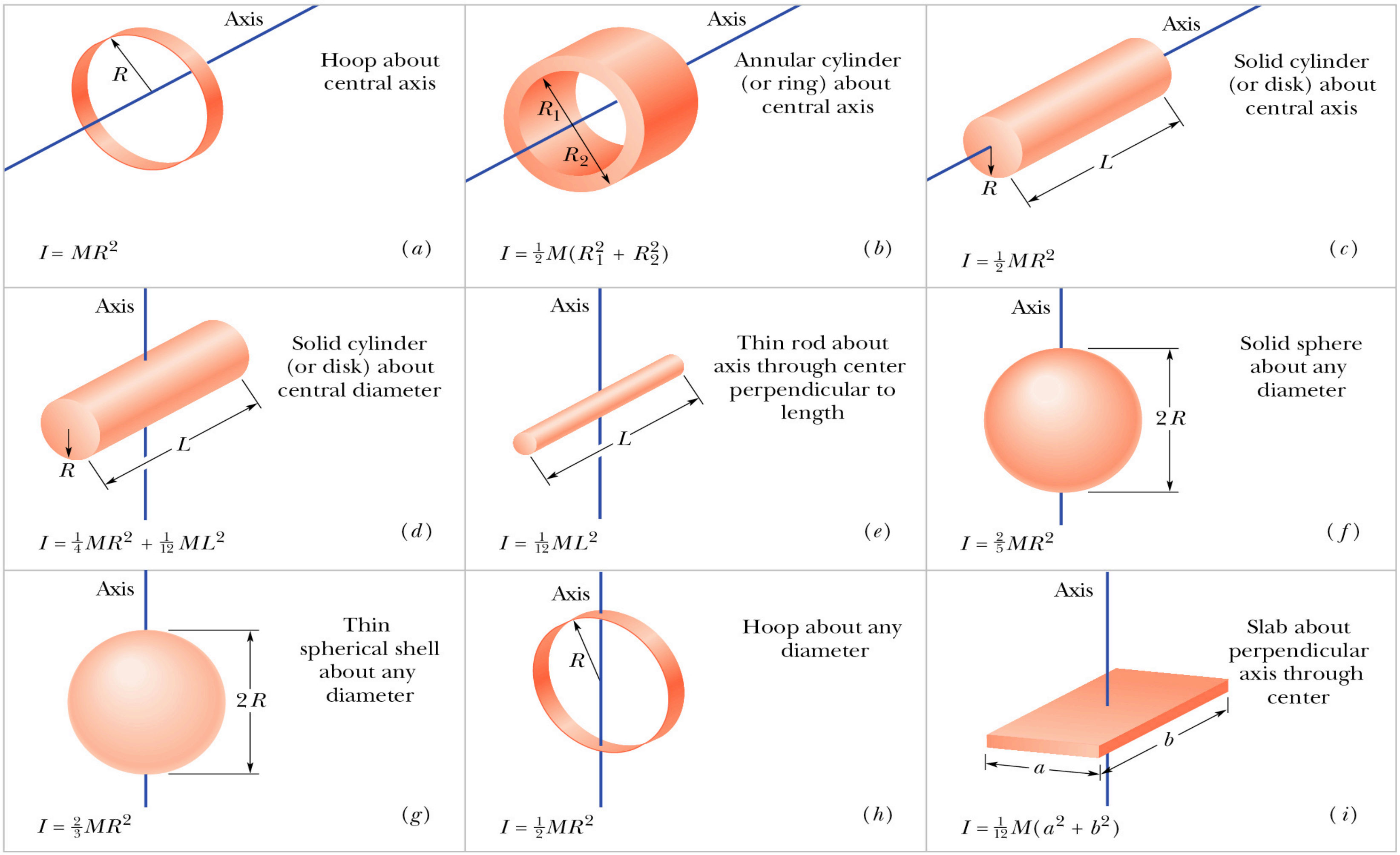
$$I = \int r^2 dm = \int_0^{2\pi} r^2 \rho r d\theta = \int_0^{2\pi} r^3 \rho d\theta$$

$$I = r^3 \rho \int_0^{2\pi} d\theta = 2\pi r^3 \rho = 2\pi r^3 \frac{M}{2\pi r} = Mr^2$$

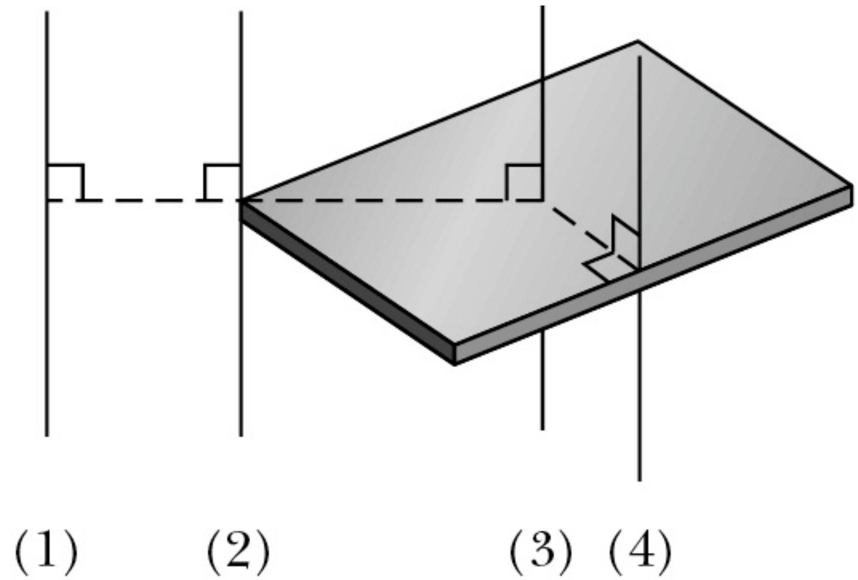
Moment of inertia

Rotational inertia involves not only the mass but also the distribution of mass for continuous masses

Calculating the rotational inertia $I = \int r^2 dm$



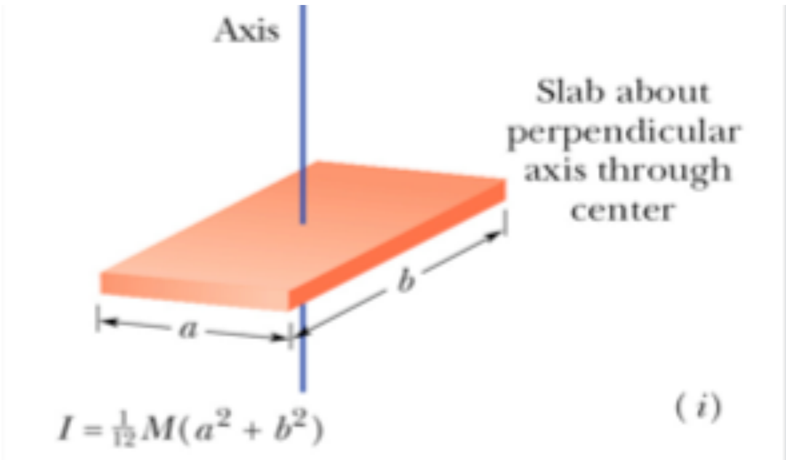
Parallel-Axis theorem

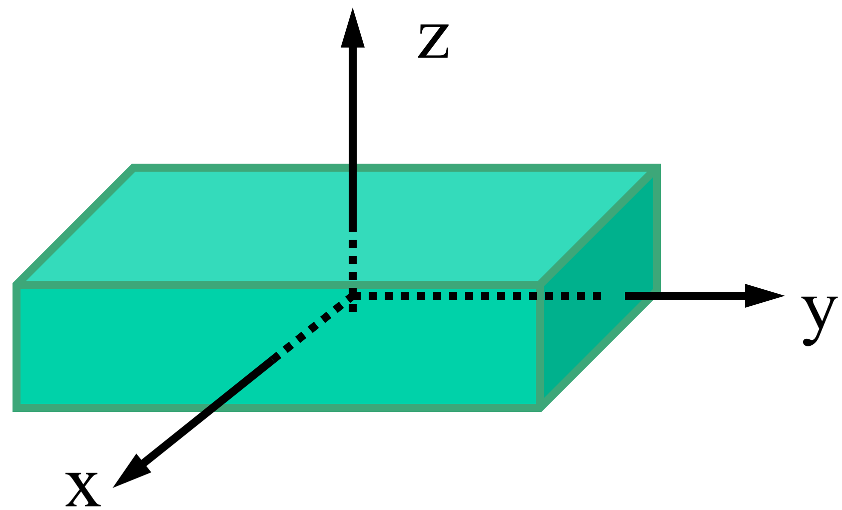


If we know the rotational inertia of a body about any axis that passes through its **center-of-mass**, we can find its rotational inertia about any other axis **parallel** to that axis with the **parallel axis theorem**

$$I = I_{c.m.} + M\,h^2$$

h: the perpendicular distance between the two axes





Consider a standard blackboard eraser. Rotation along which axis would it have the greatest moment of inertia I?

1) x
2) y
3) z
4) All have the same I

$$I = \frac{1}{12}M\left(a^2 + b^2\right)$$

Hint: Look for largest amount of mass away from the axis.

Newton's Second Law for Rotation

$$\vec{\tau}_{net} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = I\vec{\alpha}$$

I: rotational inertia

α: angular acceleration

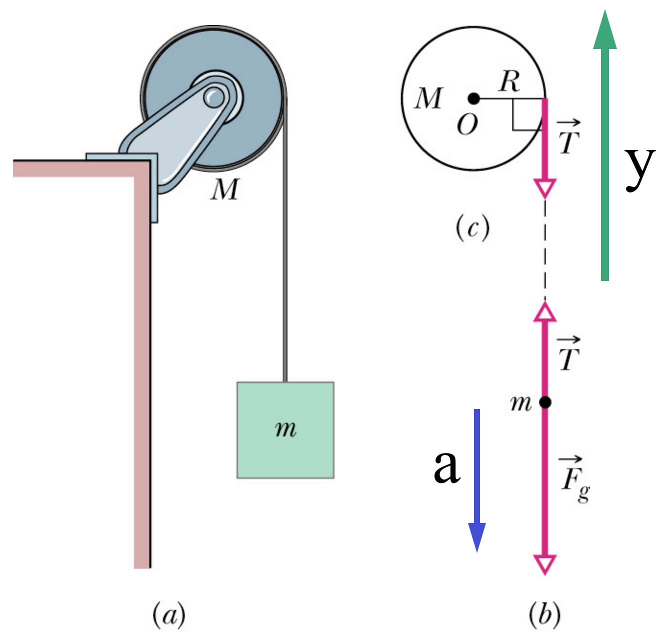
Compare to the linear equation: $\vec{F}_{net} = \sum_{i=1}^{n} \vec{F}_i = m\vec{a}$

Sample Problem 10-8

This figure shows a uniform disk, with mass M = 2.5 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk.

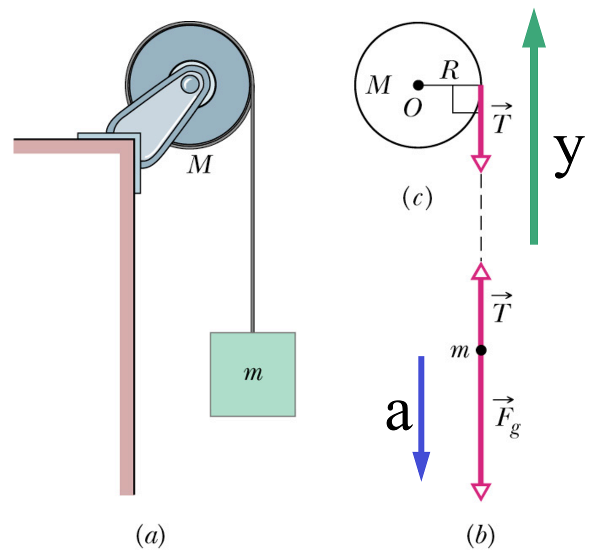
Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.

Note: R = 20 cm = 0.2 m



This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.

The key points are following:



For the block: $T - mg = ma$ (1)

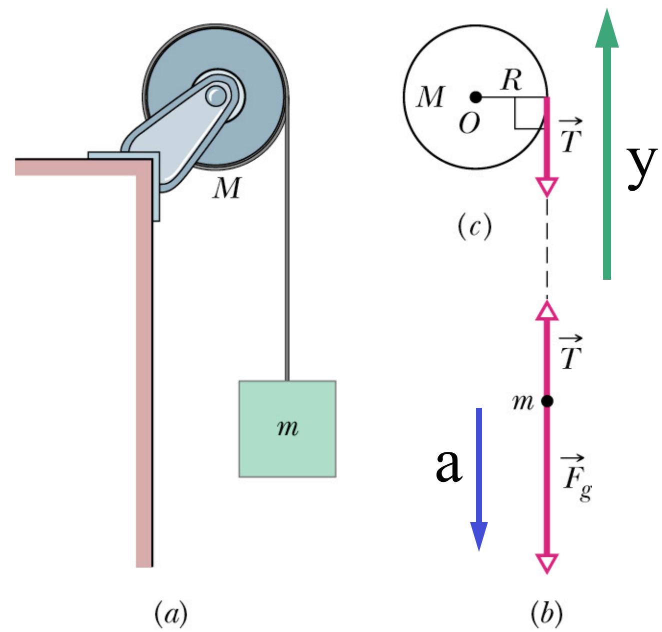
For the pulley: $\tau = I\,\alpha$

$$-RT = (1/2)MR^2\,\alpha \qquad (2)$$

acceleration a of the block is equal to a_t at the rim of the pulley $a = a_t = \alpha R$ (3)

Three equations and three unknowns: a, α, T, so a unique solution exists.

Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.



Equations 2&3:

Substitute a_t/α for R

$$T = -(1/2)Ma_t$$

Equations 2&3 and 1:

Eqn 1: $a = T/m - g$

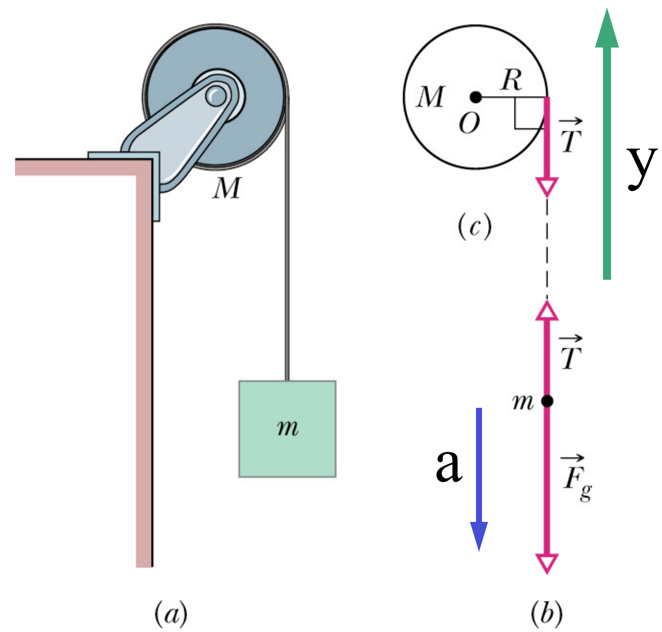
$$a = \alpha R = -(Ma)/2m - g$$

Equations 1,3 and 2:

$$a = \alpha R = g - MR\,\alpha/2m$$

$$\left(1+\frac{M}{2m}\right)a = -g \quad \Rightarrow \quad a = -\frac{2m}{(M+2m)}g = -4.8\text{m/s}^2$$

Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.



Then,

$$T = -1/2Ma = 6.0N$$

and,

$$\alpha = a/R = -24 \text{ rad/s}^2$$

Work and Rotational Kinetic Energy

Work-kinetic energy theorem: $W = \Delta K = K_f - K_i$

$\Delta K = ½ I \omega_f^2 - ½ I \omega_i^2$ (if there is only rotation)

Work done $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$ (compare to $W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$)

if τ is constant, $W = \tau(\theta_f - \theta_i)$

Power $P = dW/dt$

$P = \tau \, d\theta/dt = \tau \, \omega$

compare to $P = \vec{F} \cdot \vec{v}$

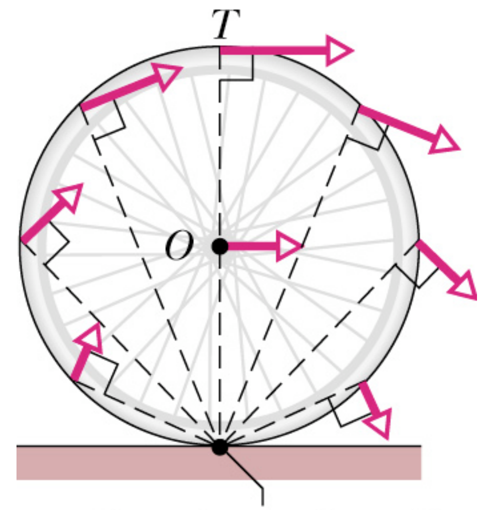
Summary -- Translation - Rotation

translational motion	Quantity	Rotational motion
x	Position	θ
Δx	Displacement	$\Delta\theta$
$v = dx/dt$	Velocity	$\omega = d\theta/dt$
$a = dv/dt$	Acceleration	α
m	Mass Inertia	I
$F = ma$	Newton's second law	$\vec{\tau} = \vec{r} \times \vec{F}$
$W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$	Work	$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$
$K = \frac{1}{2} mv^2$	Kinetic energy	$K = \frac{1}{2} I\omega^2$
$P = \vec{F} \cdot \vec{v}$	Power (constant F or t)	$P = \tau\omega$

The Kinetic Energy of Rolling

View the rolling as pure rotation around P, the kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$



Rotation axis at P

parallel axis theorem: $I_p = I_{com} + MR^2$

so $\quad K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} MR^2\omega^2$

since $\quad v_{com} = \omega R$

$$\mathbf{K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M(v_{com})^2}$$

$\frac{1}{2} I_{com}\omega^2$: due to the object's rotation about its center of mass

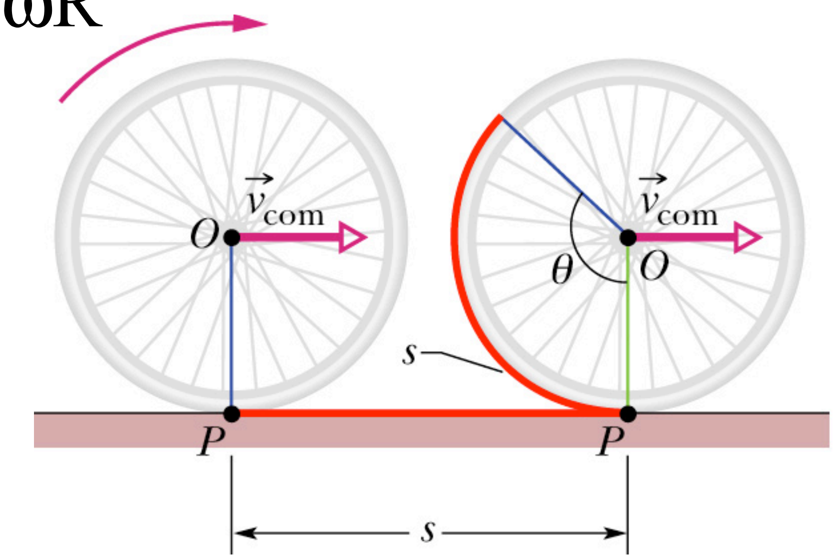
$\frac{1}{2} M(v_{com})^2$: due to the translational motion of its center of mass

Chapter 11: Rolling, Torque, and Angular Momentum

For an object rolling smoothly, the motion of the center of mass is pure translational.

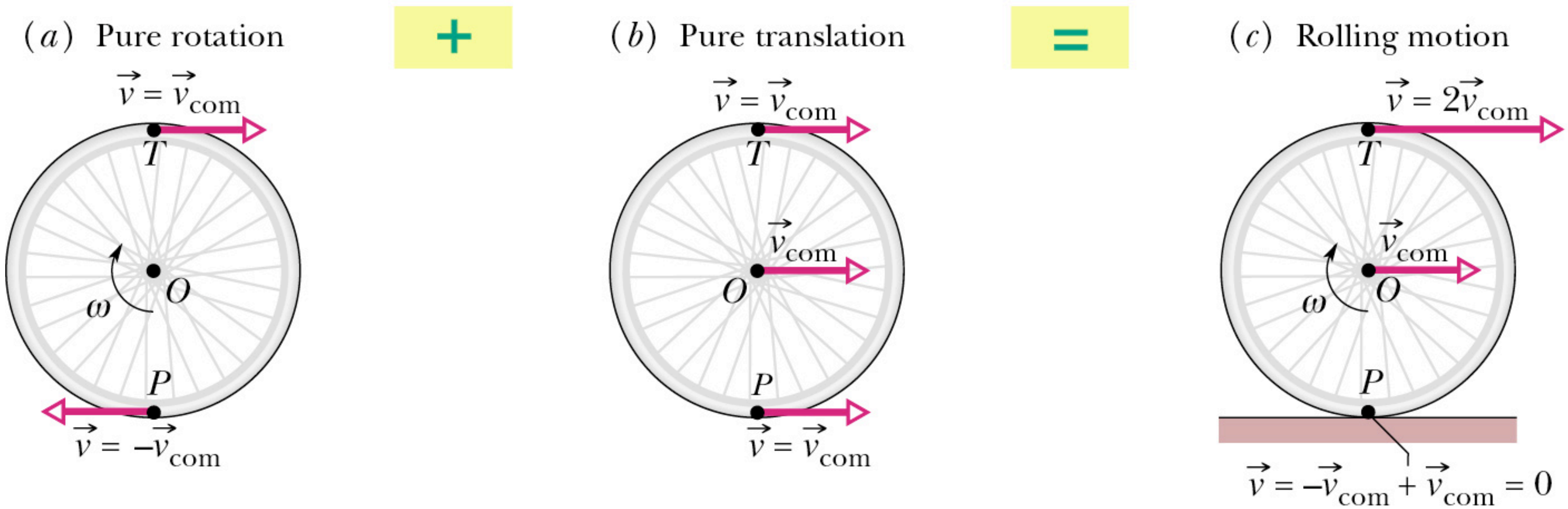
$s = \theta R$

$v_{com} = ds/dt = d(\theta R)/dt = \omega R$



$$v_{com} = \omega R$$

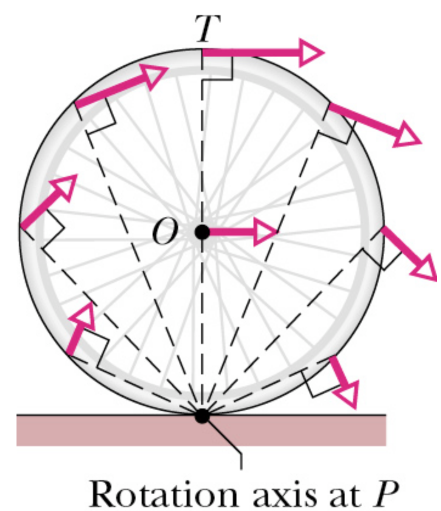
- Rolling viewed as a combination of pure rotation and pure translation



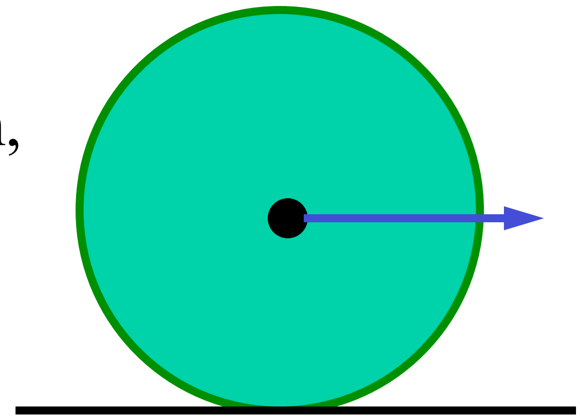
- Rolling viewed as pure rotation

  $v_{top} = (\omega)(2R) = 2\ v_{com}$

- Different views, same conclusion



Sample Problem: A uniform solid cylindrical disk, of mass M = 1.4 kg and radius R = 8.5 cm, rolls smoothly across a horizontal table at a speed of 15 cm/s. What is its kinetic energy K?



$$K = K_{trans} + K_{rot} = \frac{1}{2} M v^2_{c.m.} + \frac{1}{2} I\omega^2$$

$$v_{c.m.} = 0.15 \mathrm{m/s}$$

$$I_{disk} = 1/2 MR^2 = (0.5)(1.4\mathrm{kg})(0.085\mathrm{m})^2 = 5.058\mathrm{x}10^{-3} \mathrm{kg\ m}^2$$

$$\omega = v/R = (0.15\mathrm{m/s})/0.085\mathrm{m} = 1.765\ \mathrm{rad/s}$$

$$K = K_{trans} + K_{rot} = \frac{1}{2}(1.4)(0.15)^2 + \frac{1}{2}(5.058\mathrm{x}10^{-3})(1.765)^2$$

$$= 15.75\mathrm{x}10^{-3} + 7.878\mathrm{x}10^{-3} = 23.63\mathrm{x}10^{-3}\ \mathrm{J}$$

Angular momentum

Angular momentum with respect to point O for a particle of mass *m* and linear momentum p is defined as:

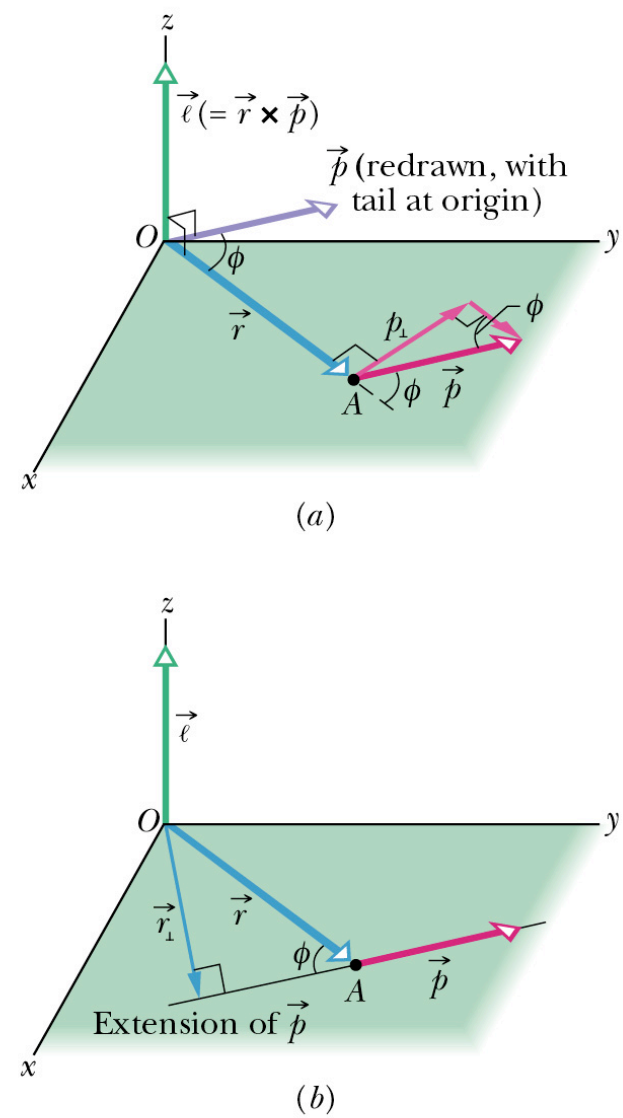
$$\boxed{\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})}$$

Compare to the linear case $\vec{p} = m\vec{v}$

direction: right-hand rule

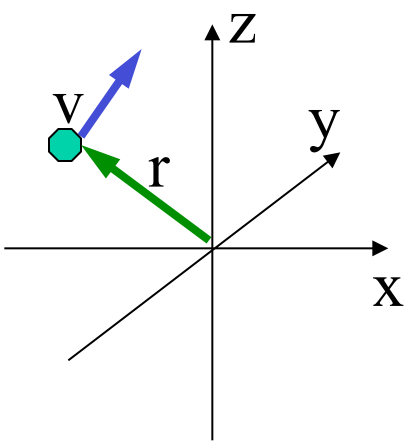
magnitude:

$$\ell = r\,p\sin\phi = r\,mv\sin\phi$$



The angular momentum of a rigid body rotating about a fixed axis

Consider a simple case, a mass *m* rotating about a fixed axis *z*:



$$\ell = r\ m\ v\ \sin 90^{o} = r\ m\ r\ \omega = mr^2\omega = I\ \omega$$

In general, the angular momentum of a rigid body rotating about a fixed axis is

$$L = I\ \omega$$

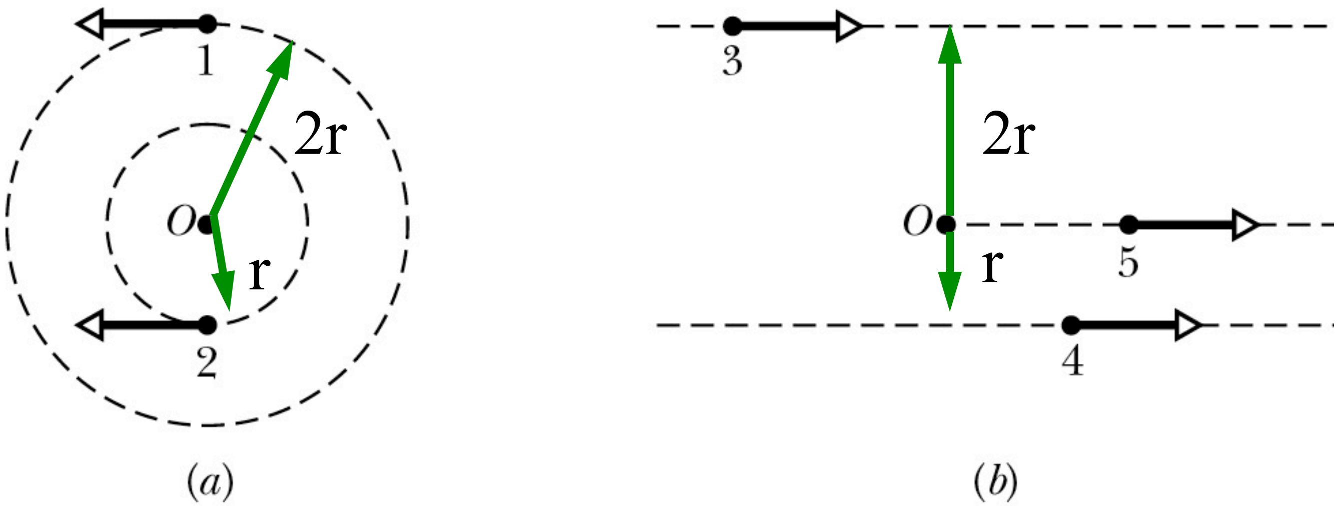
L : angular momentum (group or body) along the rotation axis

ℓ : angular momentum (particle) along the rotation axis

I : moment of inertia about the same axis

Sample Problem

Particles 1, 2, 3, 4, and 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around *O* in opposite directions. Particles 3, 4, and 5 move towards or away from *O* as shown.

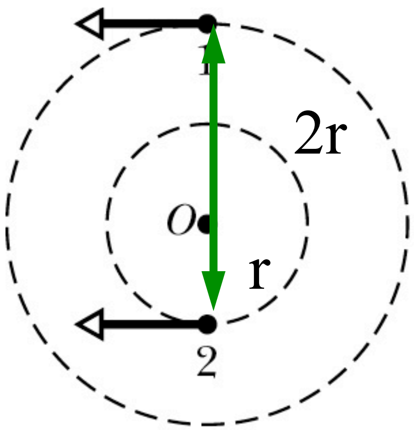


Which of the particles has the smallest magnitude angular momentum?

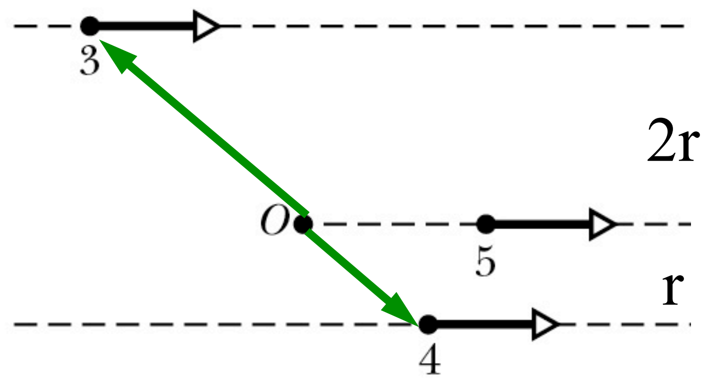
1) 1 2) 2 3) 3 4) 4 5) 5 6) all have the same l

Sample Problem

Particles 1, 2, 3, 4, and 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around *O* in opposite directions. Particles 3, 4, and 5 move towards or away from *O* as shown.



(*a*)



(*b*)

$l = r\,p\sin\phi = r\,mv\sin\phi$ $\Phi = 0^{o}$ for 5. => $l = 0$

Which of the particles has the smallest magnitude angular momentum?

1) 1 2) 2 3) 3 4) 4 5) 5 6) all have the same l

Newton's Second Law in Angular Form

$$\vec{\tau} = I\vec{\alpha} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d(m\vec{v})}{dt} = \frac{d(\vec{r} \times m\vec{v})}{dt} = \frac{d\vec{\ell}}{dt}$$

Let $\vec{\tau}_{net}$ be the vector sum of all the torques acting on the object.

$$\vec{\tau}_{net} = \frac{d\vec{L}_{total}}{dt} \qquad \vec{L}_{total} = \sum_{i=1}^{n} \vec{l}_i$$

Net external torque equals to the **time rate change** of the system's **total angular momentum**

Conservation of Angular Momentum

If the net external torque acting on a system is zero, the angular momentum of the system is conserved.

$$\text{If} \quad \vec{\tau}_{net} = 0 \quad \text{then} \quad \vec{L} = \text{constant}$$

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \qquad \vec{L}_i = \vec{L}_f$$

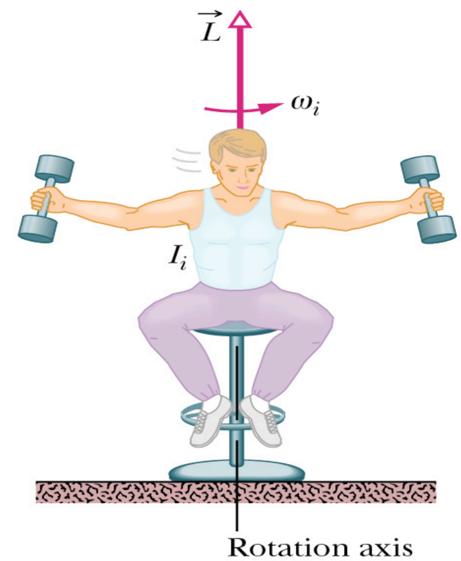
For a rigid body rotating around a fixed axis, (L = I w) the conservation of angular momentum can be written as

$$\boxed{I_i w_i = I_f w_f}$$

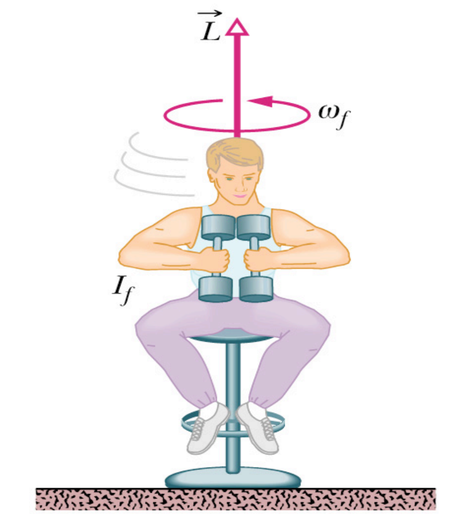
Some examples involving conservation of angular momentum

The spinning volunteer

$$\boxed{L_f = L_i \quad \Rightarrow \quad I_f w_f = I_i w_i}$$

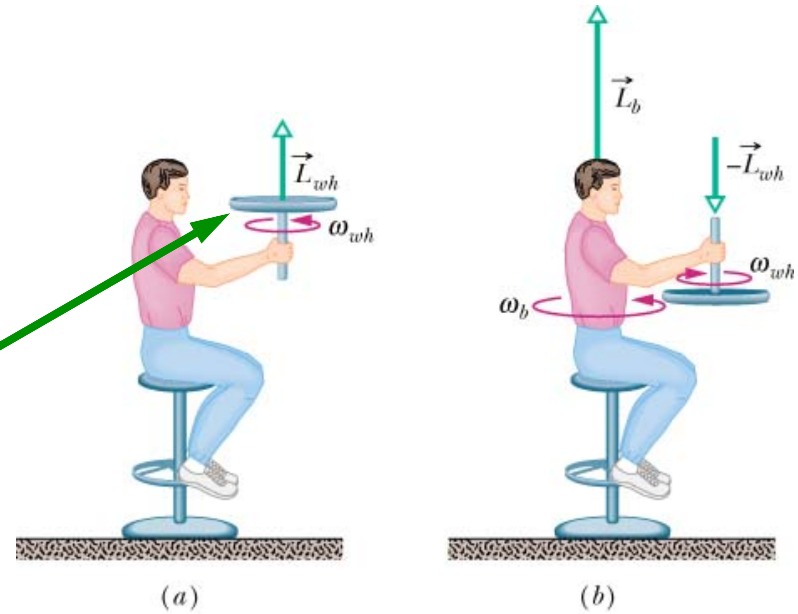


(a)



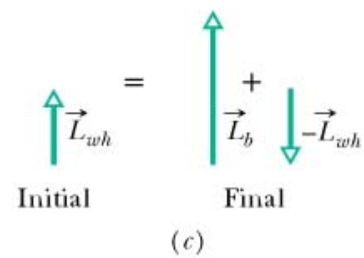
(b)

Angular momentum is conserved



L_i is in the spinning wheel

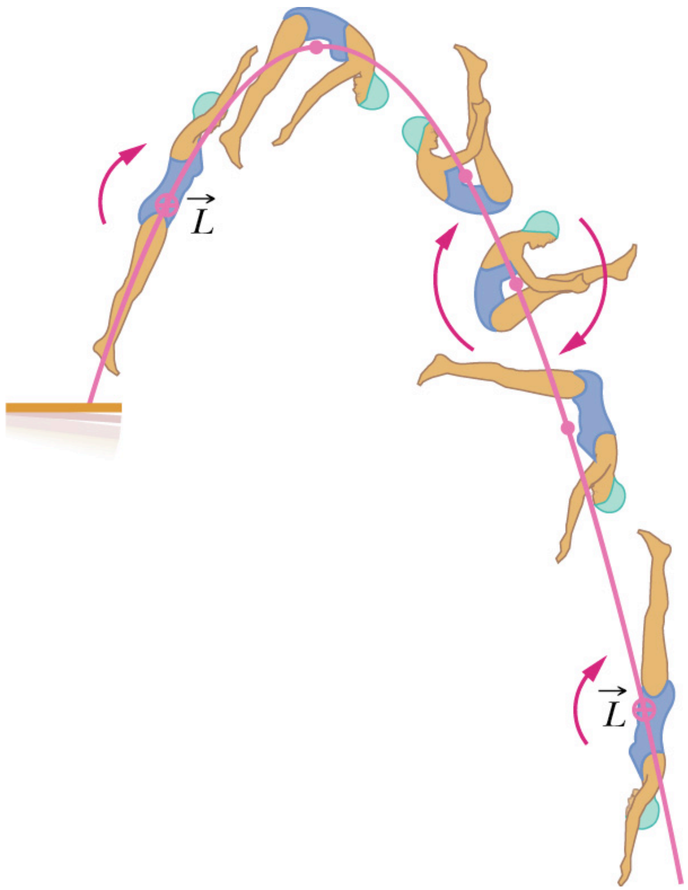
Now exert a torque to flip its rotation.
$L_{f,\ wheel} = -L_i$.



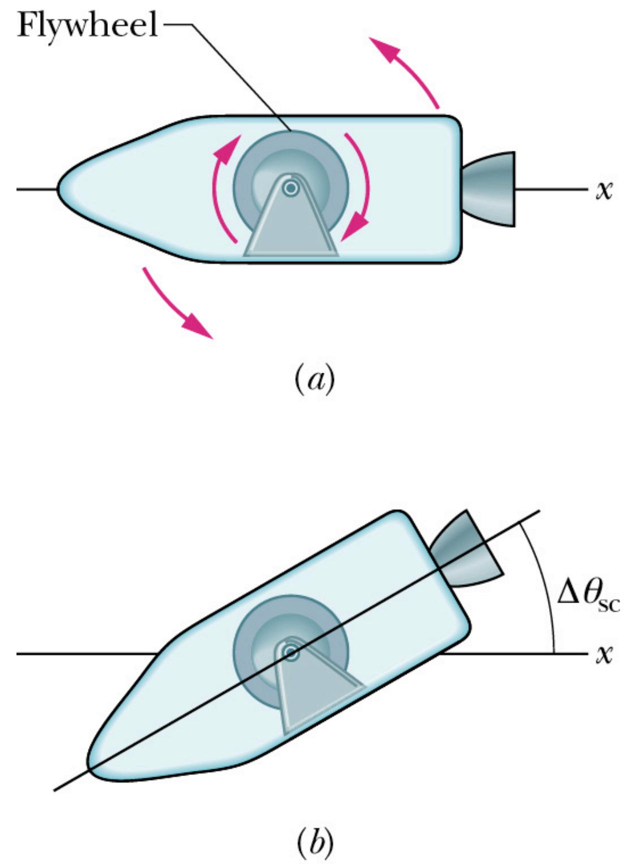
Conservation of Angular momentum means that the person must now acquire an angular momentum.
$L_{f,\ person} = +2L_i$
so that $L_f = L_{f,\ person} + L_{f,\ wheel} = +2L_i + -L_i = L_i$.

More examples

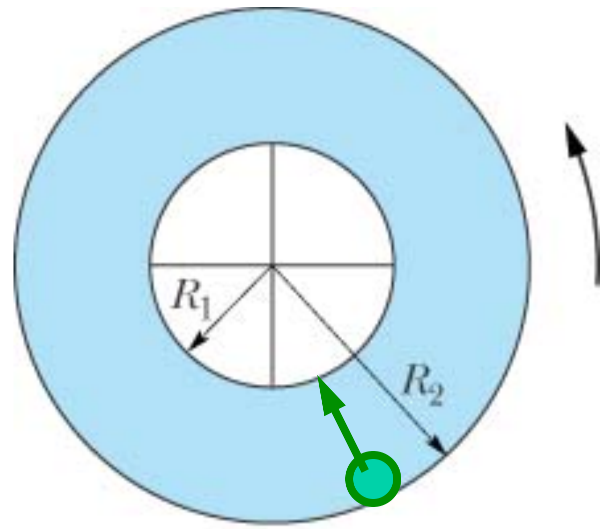
The springboard diver



Spacecraft orientation

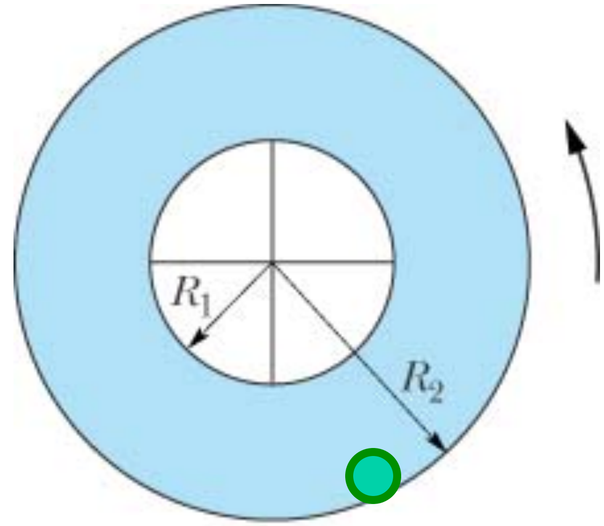


Problem 11-66



Ring of R_1 ($=R_2/2$) and R_2 ($=0.8$m),
Mass $m_2 = 8.00$kg.
$w_i = 8.00$ rad/s. Cat $m_1 = 2$kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
w_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

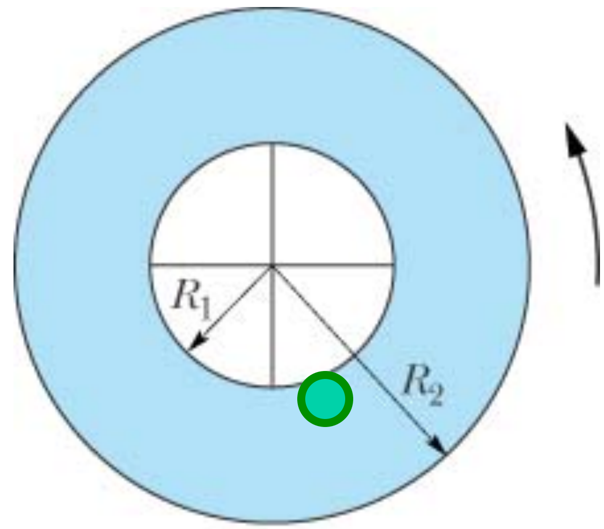
Initial Momentum

$$L_i = L_{i,cat} + L_{i,ring} = m_1 R_2 v_i + I\omega_i$$

$$= m_1 R_2^2 \omega_i + \frac{1}{2} m_2 (R_1^2 + R_2^2)\omega_i$$

$$= m_1 R_2^2 \omega_i \left(1 + \frac{1}{2} \frac{m_2}{m_1} \left(\frac{R_1^2}{R_2^2} + 1 \right) \right)$$

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
w_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Final Momentum

$$L_f = L_{f,cat} + L_{f,ring} = m_1 R_1 v_f + I\omega_f$$

$$= m_1 R_1^2 \omega_f + \frac{1}{2} m_2 (R_1^2 + R_2^2)\omega_f$$

$$= m_1 R_1^2 \omega_f \left(1 + \frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_2^2}{R_1^2} + 1\right)\right)$$

Problem 11-66

Then from $L_f = L_i$ we obtain

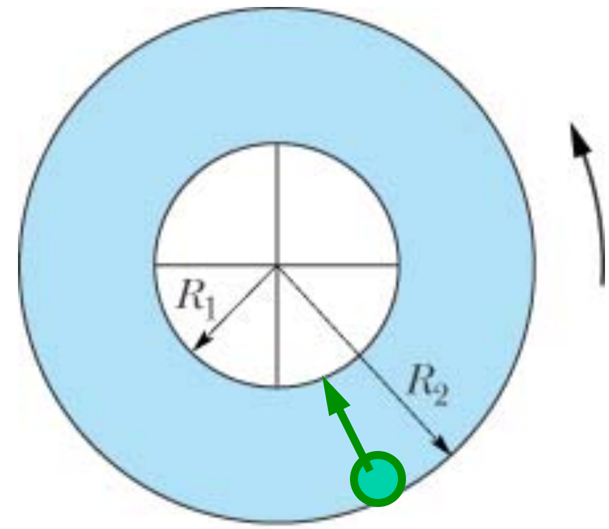
$$\frac{\omega_f}{\omega_0} = \frac{R_2^2}{R_1^2} \frac{1+\frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)}{1+\frac{1}{2}\frac{m_2}{m_1}\left(1+\frac{R_2^2}{R_1^2}\right)} = (2.0)^2 \frac{1+2(0.25+1)}{1+2(1+4)} = 1.273$$

Thus, $\omega_f = 1.273\omega_0$. Using ω_0 =8.00 rad/s, we have ω_f =10.2 rad/s. By substituting $I = L/\omega$ into $K = \frac{1}{2}I\omega^2$, we obtain $K = \frac{1}{2}L\omega$. Since $L_i = L_f$, the kinetic energy ratio becomes

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}L_f\omega_f}{\frac{1}{2}L_i\omega_i} = \frac{\omega_f}{\omega_0} = 1.273.$$

which implies $\Delta K = K_f - K_i = 0.273K_i$. The cat does positive work while walking toward the center of the ring, increasing the total kinetic energy of the system.

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
w_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Initial Kinetic energy K_i is:

$$K_i = \frac{1}{2}\left[m_1 R_2^2 + \frac{1}{2}m_2(R_1^2 + R_2^2)\right]\omega_0^2 = \frac{1}{2}m_1 R_2^2 \omega_0^2\left[1 + \frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)\right]$$

$$= \frac{1}{2}(2.00\text{ kg})(0.800\text{ m})^2(8.00\text{ rad/s})^2[1+(1/2)(4)(0.5^2+1)]$$

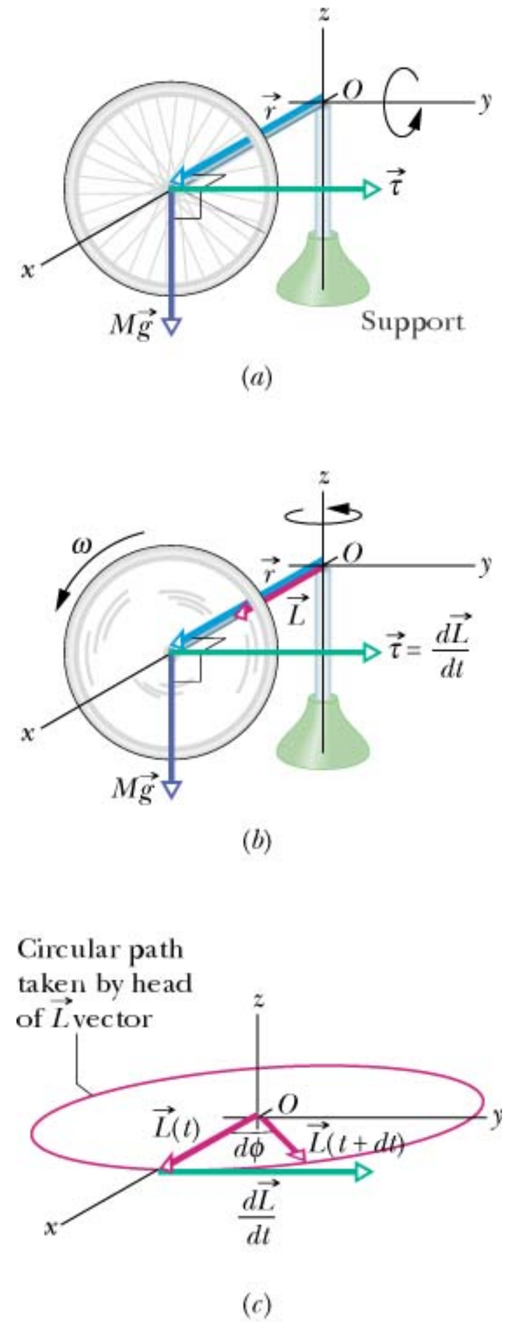
$$= 143.36\text{ J},$$

the increase in kinetic energy is $\Delta K = (0.273)(143.36\text{ J}) = 39.1\text{ J}.$

Torque and Angular Momentum

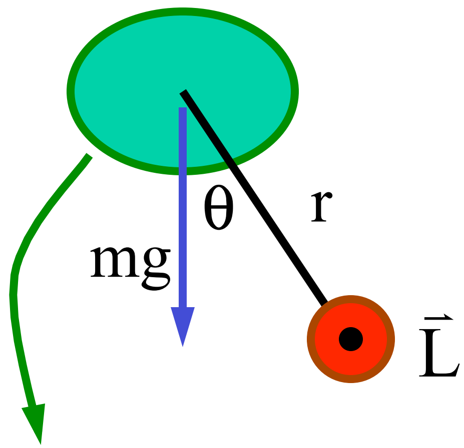
$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt} = \frac{d(\vec{r}\times m\vec{v})}{dt}$$

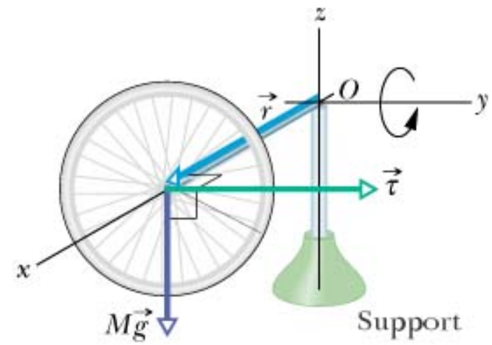
$$= \vec{r}\times\left(m\frac{d\vec{v}}{dt}\right) = \vec{r}\times(m\vec{a}) = \vec{r}\times\vec{F}$$



Torque is the time rate of change of **angular momentum**.

Precession





(a)

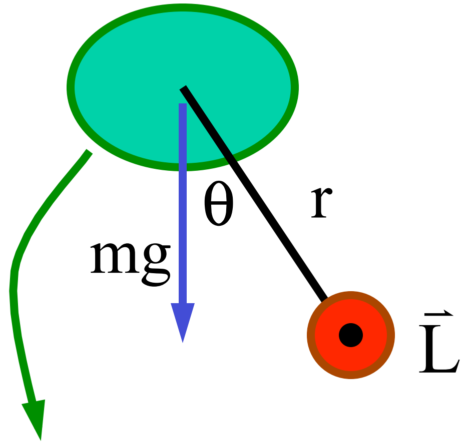
Torque is the time rate of change of angular momentum.

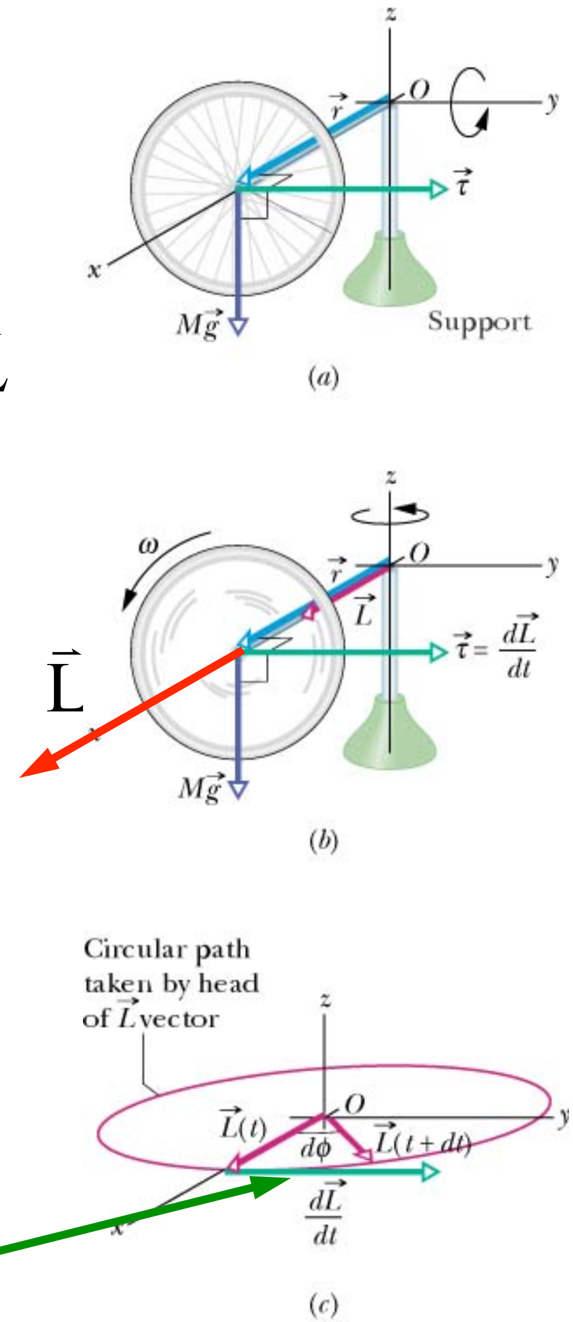
Falling due to torque about the pivot point.

$$\tau = rF\sin\theta = rmg\sin\theta$$

Falling causes angular momentum about the pivot point (along y-axis).

Precession





Torque is the time rate of change of angular momentum.

Falling due to torque about the pivot point.

$\tau = rF\sin\theta = rmg\sin\theta$

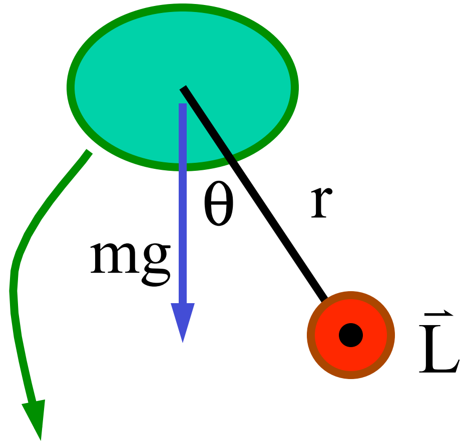
Falling causes angular momentum about the pivot point (along y-axis).

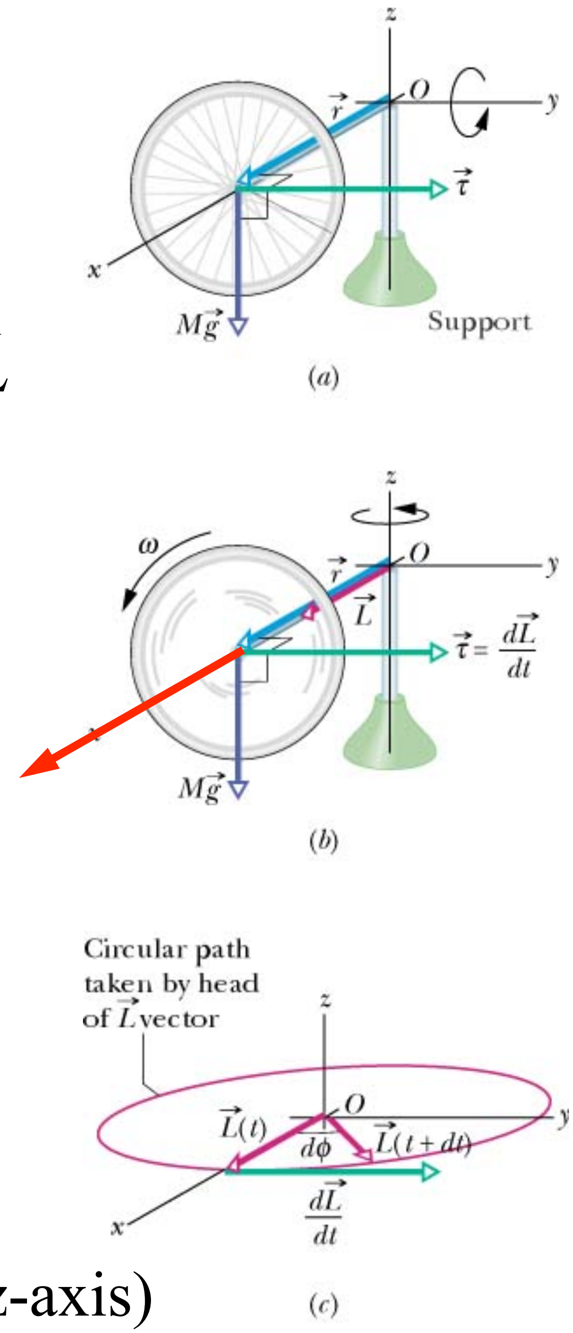
Now set the gyroscope in motion

$L = Iw$ (along x-axis)

L is fixed by the spinning, so the torque can only change the direction of L

Precession Rate





Torque is the time rate of change of angular momentum.

Falling due to torque about the pivot point.

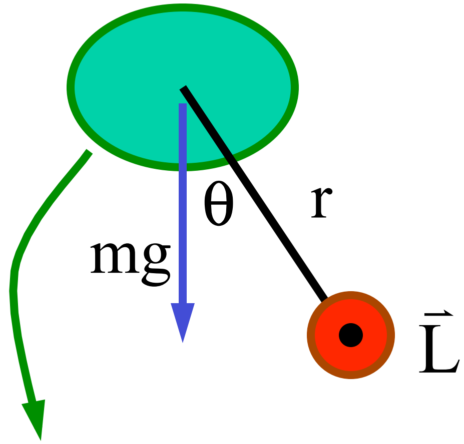
Falling causes angular momentum about the pivot point (along y-axis).

$$d\phi = \frac{dL}{L} = \frac{Mg\,r\,dt}{I\omega}$$

$$\Omega \equiv \frac{d\phi}{dt} = \frac{Mg\,r}{I\omega}$$

(Precession along z-axis)

Precession Rate



<u>Torque</u> is the time rate of change of <u>angular momentum</u>.

Nuclei have intrinsic angular momentum.

This effect is at the core of MRI, which is tuned to pick up the intrinsic angular momentum of the proton in hydrogen.

(Precession along z-axis)