As another example, we have previously developed geodesic clock reference systems that are equivalent to the well known infalling and rising Eddington–Finkelstein coordinate systems.[4] These reference systems consist of a succession of clocks measuring time τ^* that drop from, or rise to, some constant radius $R_c > R_s$. Each clock is synchronized such that the time recorded by it as it drops from, or rises to, R_c equals the time on a clock that remains fixed at R_c. Thus we have a picture of a sort of clock factory fixed at R_c, that at regular intervals drops clocks that are synchronized with a master clock in the clock factory.

The transformation that replaces the Schwarzschild time coordinate T with the geodesic clock time coordinate τ^* has the same form as Eq. (2.4), except the variable quantity R_i is replaced with the constant quantity R_c. The resulting nondiagonal metric has the form

$$ds^2(R,\tau^*) = \frac{1}{1-(R_s/R_c)}\left[dR - m\sqrt{\frac{R_s}{R} - \frac{R_s}{R_c}}\,d\tau^*\right]^2 - d\tau^{*2} + R^2\,d\Omega^2, \quad m=\pm 1, \qquad \text{(B1)}$$

where $m=-1(+1)$ for clocks that fall from (rise to) R_c. This metric form looks like the metric form (2.1), except here R_c is a constant and not a variable.

Figures 5(a) and 5(b) show (R,τ^*) spacetime diagrams where time is, respectively, measured with infalling clocks $(m=-1)$ and rising clocks $(m=+1)$. It is seen that all the world lines of all the previous figures are repeated in a one-to-one fashion, including the limiting timelike geodesic dd and the timelike trajectories in the shaded regions that cannot be followed to $R > R_c$. The familiar "tilting" of light cones in Eddington–Finkelstein coordinates is also seen in the coordinates (R,τ^*).

The τ^*-reference system could be straightened out by introducing a new comoving spatial coordinate that stays constant along the world line of each geodesic clock. This would result in the nondiagonal metric form (B1) being changed into a diagonal form. But there is nothing to be gained from this.

[1]A discussion and comparison of the coordinate system of Eddington–Finkelstein, Kruskal–Szekeres, and Novikov, including references to the original and related papers, can be found in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Secs. 31.4 and 31.5.

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The approximation to the exponential decay law

H. Jakobovits
Service de Chimie Physique, Faculte des Sciences, Universite Libre de Bruxelles, 1050 Brussels, Belgium

Yehuda Rothschild
Department of Physics, Bar-Ilan University, Ramat-Gan 52100, Israel

J. Levitan[a)]
Department of Physics, College of Judea and Samaria and International Solvay Institutes for Physics and Chemistry, Universite Libre de Bruxelles, 1050 Brussels, Belgium

(Received 14 February 1994; accepted 26 October 1994)

The exponential decay law is a consequence of the choice of the "Breit–Wigner" energy distribution when the integration over the energy is extended from $-\infty$ to $+\infty$. We explain why the Breit–Wigner function is the natural choice and study by the method of steepest descent and by direct integration how an integration over the physically relevant positive energy spectrum affects the decay law. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Almost all known elementary particles are unstable and decay according to the exponential decay law.[1]

The empirical support for the decay law, which even at high school level can easily be demonstrated with the help of a Geiger counter, is so convincing that one often tends to neglect the fact that the decay law in a quantum mechanical treatment is not exactly exponential and that the approach to the exponential is a result of delicate approximations.[2]

The quantum mechanical treatment of a decaying system was established by Wigner and Weisskopf.[3] One chooses an element ϕ of the Hilbert space, usually identified as the eigenfunction of some unperturbed Hamiltonian H_0, as the initial state of the system. The full Hamiltonian, $H=H_0+V$, where V is a perturbation under which the original system is unstable, generates an evolution of ϕ, which is understood as the "decay" of this unstable state, i.e.,

$$\psi_t = e^{-iHt}\phi.$$

Here, e^{-iHt} is the one-parameter group (which satisfies $e^{-iHt1}e^{-iHt2} = e^{-iH(t_1+t_2)}$ of the Schrödinger motion; clearly ψ_t satisfies (we take $\hbar=1$)

$$i\frac{\partial\psi}{\partial t}=H\psi_t.$$

The decay law is based on the assumption that the square of the survival amplitude[4]

$$A(t)=\langle\phi|e^{-iHt}|\phi\rangle,$$

is the probability $P(t)$ that the system has not decayed.

Wigner and Weisskopf[3] showed that this probability is approximately exponential in form for t not too large and not too small. It is clear that, for very small t, the decay law cannot be exponential, since for Hermitian H (and when $H|\phi\rangle$ is defined) (see also Ref. 8)

$$\frac{d}{dt}[A(t)]^2|_{t=0}=0.$$

The time before passing to an approximate exponential behavior is generally (for small coupling) very small compared to the lifetime.[1,4]

An exact exponential form in the quantum theory is obtained by taking the Fourier transform of the Breit–Wigner energy density distribution (given below) from $-\infty$ to $+\infty$. The fact that the actual energy spectrum is bounded from below leads (via use of the Paley–Wiener theorem) to a deviation from the exponential for large times.[4] It has recently been shown,[5] in fact, that for an unstable quantum system for which the unperturbed Hamiltonian has discrete states embedded in a continuous spectrum on $(-\infty,\infty)$, the time dependence of the decay is a sum of exponential contributions plus a background contribution which may be arbitrarily small for any positive t.

For small times it is the expectation value of $(H-\langle H\rangle)^2$ in the initial state $|\phi\rangle$ which leads to a measure of the deviation from the pure exponential.[6,7] (Note that the boundness of the Hamiltonian on $|\phi\rangle$ also is crucial for the short time deviation.)

In many situations it is difficult to estimate the contribution of the integration from $-\infty$ to zero and in this way evaluate the error when one neglects the semiboundness and instead integrates over an energy spectrum covering all the real axis. We show, however, by the use of the method of the steepest descent, that one can demonstrate in a simple and elegant way that the exponential decay law is a good approximation for the decay law in many interesting cases. The use of the method of the steepest descent, which carries pedagogical value as well, has to our knowledge, never been exploited in this framework.

The energy distribution function $\omega(E)$ is defined as the Fourier transform of the nondecay amplitude $A(t)$, i.e.,

$$A(t)=\int_0^\infty e^{-iEt}\omega(E)dE. \tag{1}$$

We see that $\omega(E)$ is the density, or *a priori* amplitude with which the component e^{-iEt} occurs. Hence, $\omega(E)$ is called the *energy density distribution.*

The nondecay probability, $P(t)$, is then determined, provided the energy density distribution, $\omega(E)$, is specified.[8] The integral distribution function $W(E)$ is related to $\omega(E)$ by

$$W(E')-W(E)=\int_{E'}^{E}\omega(E)dE,$$

where E and E' are any two values of energy. Krylov and Fock[9] concluded that a system decays if and only if the integral energy distribution function is continuous.

An exponential decay law has been derived by Khalfin, as an approximation to the actual time development, by using the Breit–Wigner (BW) energy density distribution, and by extending the integral in Eq. (1) from $-\infty$ to ∞. Thus

$$A(t)\approx\int_{-\infty}^{\infty}e^{-iEt}\frac{\Gamma}{\pi[(E-E_0)^2+\Gamma^2]}dE. \tag{2}$$

When studying exponential decay, the energy density distribution is often chosen to be the BW function,

$$\omega(E)=\frac{\Gamma}{\pi[(E-E_0)^2+\Gamma^2]}.$$

It is chosen to reflect the exponential decay law upon integration from $-\infty$ to ∞ with e^{-iEt}, as we discuss in the next section. One could, however, ask if the energy density distribution is uniquely determined by the decay law,

$$P(t)=e^{-2\Gamma t}.$$

We shall show that this is not the case, but that given this decay law, the BW function must appear in the energy density distribution, and that is why it has so often been typically chosen.

We then examine another observed decay law, that given by the product of an exponential and a polynomial. A simple case for which such decay laws can occur has been investigated by Stodolsky.[10]

We find the energy distribution density which, when allowing the energy to be integrated from $-\infty$ to ∞, gives rise to this more general type of decay. When, however, the energy is semi bounded, that is, when the energy distribution density is nonzero only for positive values of E, we derive a different decay law using direct integration. Using the "method of steepest descent," we show that the exponential term is the leading term, and hence the exponential decay law is a good approximation for a decay which emerges from the BW energy density distribution. The result of the direct integration from 0 to ∞ is a decay law, given by an exponential term and a polynomial with negative powers of t.

II. EXPONENTIAL DECAY: WHY THE BW ENERGY DENSITY DISTRIBUTION?

We will first show how the BW energy density distribution gives rise to approximate exponential decay.

We calculate the decay law explicitly, carrying out integral (2). Considering E as a complex variable, the integrand has poles at $E=E_0\pm i\Gamma$. The integral can be calculated using complex contour integration to obtain

$$A(t)=e^{-iE_0t}\cdot e^{-\Gamma t}. \tag{3}$$

We see that the important term in $A(t)$, which is $e^{-\Gamma t}$, comes from the residue of e^{-iEt} at the pole in the lower half-plane of E.

We now let $\omega(E)$ be a general real and positive energy density distribution. We shall try to reconstruct $\omega(E)$ based on the following considerations. In order to have exponential decay, $\omega(E)$ must have a pole in the lower half-plane. The pole should not be too close to the real axis because as Γ approaches 0, $e^{-\Gamma t}$ approaches $e^{-0t}=1$, i.e., there is no decay. This corresponds to the fact that the lifetime is proportional to $1/\Gamma$. The presence of a pole at $(E_0-i\Gamma)$ implies that $\omega(E)$ contains a factor $1/(E-E_0+i\Gamma)$. In addition, in order to have physical interpretation, $\omega(E)$ must be real for all real E. The complex conjugate of the first factor,

$1/(E-E_0-i\Gamma)$, must therefore also be a factor of $\omega(E)$. The residue of this factor at the negative pole results in a Γ factor in the denominator. In order not to have any inverse Γ dependence in the decay law, therefore, we need a Γ in the numerator of $\omega(E)$ in order to cancel the Γ in the denominator. This means that $\omega(E)$ must be of the form

$$\omega(E)=\frac{\Gamma}{[E-(E_0-i\Gamma)][E-(E_0+i\Gamma)]}\cdot f(E)$$
$$=\frac{\Gamma}{[(E-E_0)^2+\Gamma^2]}\cdot f(E).$$

If $f(E)$ has at least one pole then, since $f(E)$ is real for all real E, it has at least two poles, one in the upper half-plane and one in the lower half-plane. We consider the case where $f(E)$ has a pole different from $E_0\pm i\Gamma$. The decay law will then be given by the sum of the residue of $\omega(E)$ at the pole $E_0-i\Gamma$ and the residue at the pole (or poles) of $f(E)$ which lie in the lower half-plane. This will result in a decay law which is the sum of two or more exponentials

$$P(t)=e^{-2\Gamma_1 t}+e^{-2\Gamma_2 t}+\cdots .$$

The long-time behavior of the system will be dominated by the pole Γ_i which is closest to the real axis

$$P(t)_{t-\infty}\rightarrow e^{-2\Gamma_1 t}.$$

If we are interested in pure exponential decay, then this case is ruled out, and $f(E)$ can have no poles.

It was shown by Krylov and Fock[9] that the energy density distribution $\omega(E)$ of the decaying state exhibits various properties. In particular, in the problem of the emission of a particle from a well through a potential energy barrier (α decay) the energy density distribution $\omega(E)$ is a meromorphic function of the complex variable E. Since we have said that $f(E)$ has no poles, if we assume that we are dealing with only meromorphic functions, then $f(E)$ must be an entire function. In addition, we suppose that $f(E_0-i\Gamma)$ has modulus 1. We have thus shown that $\omega(E)$ is the product of a BW function and an analytic function $f(E)$ such that $f(E)$ is real for all real E, and $|f(E_0-i\Gamma)|=1$. Since there are many such functions, $\omega(E)$ is not uniquely determined.

We would not expect $\omega(E)$ to be unique, since $\omega(E)$ is given by the Fourier transform of $A(t)$, where the modulus of $A(t)$ is $e^{-\Gamma t}$ but where the phase factor of $A(t)$ is unknown

$$\omega(E)=\mathscr{F}^{-1}(e^{-\Gamma|t|}\cdot e^{i\phi(t)}),$$

where $\phi(t)$ is arbitrary. When $\phi(t)$ is linear in t, the Fourier transform gives the BW distribution. But if $\phi(t)$ is quadratic or some other nonlinear function, the inverse Fourier transform will be different from the BW function. A study of the effects of various distortions of the BW energy density distribution has been done by Chiu, Misra, and Sudarshan.[11]

III. EXPONENTIAL TIMES POLYNOMIAL DECAY: WHICH ENERGY DENSITY DISTRIBUTIONS GIVE RISE TO THIS TYPE OF DECAY?

We let $A(t)$ take the most general form representing exponentials multiplied by polynomials with positive powers of t

$$A(t)=\sum_{k=0}^{m}\sum_{j=0}^{n} C_{jk}|t|^j e^{-(E_k-i\Gamma_k)|t|}. \tag{4}$$

To find an energy density distribution that could give rise to this type of decay, we take the inverse Fourier transform of $A(t)$

$$\omega(E)=\mathscr{F}^{-1}(A(t))=F^{-1}\left(\sum_{k=0}^{m}\sum_{j=0}^{n} C_{jk}|t_j|e^{-\Gamma_k|t|}e^{-iE_k|t|}\right)$$
$$=\sum_{k=0}^{m}\sum_{j=0}^{n} C_{kj}\int_{-\infty}^{\infty}|t|^j e^{-\Gamma_k|t|}e^{-iE_k|t|}e^{iE|t|}\,dt$$
$$=\sum_{k=0}^{m}\sum_{j=0}^{n} C_{kj}2_0\int_0^{\infty}t^j e^{-\Gamma_k t}\cos(ut)dt$$
$$=\sum_{k=0}^{m}\int_0^{\infty}\sum_{l=0}^{p} a_{kl}t^{l+1/2}K_{l+1/2}(\Gamma_k t)\cos(ut)dt$$
$$=\sum_{k=0}^{m}\sum_{l=0}^{p} a_{kl}\frac{1}{(u^2+\Gamma_k^2)^{l+1}}$$
$$=\sum_{k=0}^{m}\sum_{l=0}^{p} a_{kl}\frac{1}{[(E-E_k)^2+\Gamma_k^2]^{l+1}}. \tag{5}$$

We shall estimate the difference between the integration over the whole real line of the energy variable E and the positive real line. Since the difference between the two which contributes significantly to the integral is a region of small energy, we expect any difference in the decay to appear for large times. We are therefore interested in determining the large-time behavior. The integration over the whole real line results in the general form for $A(t)$ which is given in Eq. (4).

Calculation of $A(t)$ from semibounded spectrum. If the spectrum is limited to $(0,\infty)$ we must use approximation methods to calculate the decay.

A. Direct integration

We now calculate $A_{kl}(t)$

$$A_{kl}(t)=\frac{1}{\pi}\int_0^{\infty}\frac{a_{kl}e^{-iE_k t}}{[(E-E_k)^2+\Gamma_k^2]^{l+1}}dE$$
$$=\frac{1}{\pi}\int_0^{\infty}e^{-iE_j t}\frac{a_{kl}\cos(ut)}{[u^2+\Gamma_k^2]^{l+1}}du$$
$$-\frac{i}{\pi}\int_0^{\infty}e^{-iE_k t}\frac{a_{kl}\sin(ut)}{(u^2+\Gamma_k^2)^{j+1}}$$
$$+\frac{1}{\pi}\int_{-E_k}^{0}e^{-iE_k t}\frac{a_{kl}e^{-iut}}{(u^2+\Gamma_k^2)^{l+1}}du$$
$$=\frac{1}{\sqrt{\pi}}a_{kl}e^{-iEt}\left(\frac{t}{2\Gamma_k}\right)^{l+1/2}K_{l+1/2}(\Gamma_k t)/\Gamma(l+1)$$
$$-\frac{ia_{kl}}{\pi t[E_k^2+\Gamma_k^2]^{l+1}}-\frac{2a_{kl}E_k}{\pi t^2(E_k^2+\Gamma_k^2)^{l+2}}\cdots,$$

where $K_{l+1/2}(\Gamma_k t)$ is a modified Bessel function of imaginary argument. For example, some of the terms are as follows:

$$A_{00}(t)=a_{00}e^{-iE_0t}e^{-\Gamma_0t}\cdot\frac{1}{2\Gamma_0}-\frac{ia_{00}}{\pi t(E_0^2+\Gamma_0^2)}-\frac{2a_{00}E_0}{\pi t^2(E_0^2+\Gamma_0^2)^2}, \tag{6a}$$

$$A_{11}(t)=a_{11}e^{-iE_1t}e^{-\Gamma_1t}(1+\Gamma_1t)\frac{1}{4\Gamma_1^3}-\frac{ia_{11}}{\pi t(E_1^2+\Gamma_1^2)^2}-\frac{2a_{11}E_1}{\pi t^2(E_1^2+\Gamma_1^2)^3}, \tag{6b}$$

$$A_{12}(t)=a_{12}e^{-iE_2t}e^{-\Gamma_2t}(1+\Gamma_2t)\frac{1}{4\Gamma_2^3}-\frac{ia_{12}}{\pi t(E_2^2+\Gamma_2^2)^2}-\frac{2a_{12}E_2}{\pi t^2(E_2^2+\Gamma_2^2)^3}, \tag{6c}$$

$$A_{22}(t)=a_{22}e^{-iE_2t}e^{-\Gamma_2t}(3+3\Gamma_2t+\Gamma_2t^2)\frac{1}{16\Gamma_2^5}-\frac{ia_{22}}{\pi t(E_2^2-\Gamma_2^2)^3}-\frac{2a_{22}E_2}{\pi t^2(E_2^2+\Gamma_2^2)^4}. \tag{6d}$$

By only retaining the first term in the expansion for $A_{00}(t)$, the well known expression for $A(t)$, which can be found in any textbook on decaying systems, is recovered. The inclusion of $A_{11}(t), A_{12}(t)$, etc., would lead to a distortion of the BW distribution and hence imply a different decay law.

In order to get a significant contribution from the second term in $A_\infty(t)$, one must have $\Gamma_0\approx E_0$ which is only fulfilled for a very short lived (broad) state. In order that the expansion should be valid one must demand that $t\gg 1/E_0$ [in fact, $t=1/E_0$ implies that the second term and the third in the expansion of $A_{00}(t)$ are identical].

B. The method of the steepest descent

We now apply the method of steepest descent to calculate $A_0(t)$. We first rewrite $A_0(t)$ in the following form:

$$A_0(t)=\frac{1}{\pi}\Gamma t\int_0^\infty d\sigma\int_0^\infty dE\; e^{-iEt}e^{-\sigma t[(E-E_0)^2+\Gamma^2]}=\frac{1}{\pi}\Gamma t\int_0^\infty d\sigma\int_0^\infty e^{tf(z)}g(z)dz, \tag{7}$$

where

$$f(z)=-iz-\sigma[(z-E_0)^2+\Gamma^2]g(z)=1, \tag{8}$$

$$f'(z)=-i-2\sigma(z-E_0)=0 \quad \text{for } z_0=E_0-\frac{i}{2\sigma}, \tag{9}$$

$$f''(z)=-2\sigma. \tag{10}$$

We have no higher derivatives of $f(z)$ different from zero and $g(z)\equiv 1$ so the saddle point approximation contains just one term. We now can approximate the second integral in Eq. (7)

$$\int_0^\infty e^{tf(z)}g(z)dz\simeq\frac{\sqrt{2\pi}g(z_0)}{\sqrt{t|f''(z_0)|}}e^{tf(z_0)}e^{i\alpha}.$$

And hence

$$=\frac{\sqrt{2\pi}}{\sqrt{2\sigma t}}e^{t(-iE_0-1/2\sigma+1/4\sigma-\sigma\Gamma^2)}, \tag{11}$$

$$A_0(t)=\frac{1}{\pi}\Gamma t\int_0^\infty\sqrt{\frac{\pi}{t\sigma}}e^{t(-iE_0-1/4\sigma-\sigma\Gamma^2)},$$

$$=\Gamma\sqrt{\frac{t}{\pi}}e^{-iE_0t}\int_0^\infty d\sigma\frac{e^{-t(1/4\Gamma^2\sigma+1/4\sigma)}}{\sqrt{\sigma}},$$

$$p=2\sigma\Gamma,\quad \sigma=\frac{p}{2\Gamma},\quad dp=2\Gamma\, d\sigma. \tag{12}$$

We therefore find that Eq. (7) becomes

$$A_0(t)=\Gamma\sqrt{\frac{t}{\pi}}e^{-iE_0t}\int_0^\infty\frac{1}{\sqrt{2\Gamma}}dP\frac{e^{-t(\Gamma/2P+\Gamma P/2)}}{\sqrt{P}}$$

$$=\Gamma\sqrt{\frac{t}{\pi}\frac{e^{-iE_0t}}{\sqrt{2\Gamma}}}\int_0^\infty dP\frac{e^{-\Gamma t/2(P+1/P)}}{\sqrt{P}}$$

$$=\Gamma\sqrt{\frac{t}{\pi}\frac{e^{-iE_0t}}{\sqrt{2\Gamma}}}K_{1/2}(\Gamma t), \tag{13}$$

where K is the modified Bessel function.

$$K_\nu(x)=\int_0^\infty ds\frac{e^{-x/2(s+1/s)}}{S^{1-\nu}},$$

$$K_\nu(x)=\frac{\pi}{2\sin\nu\pi}\{I_{-\nu}(x)-I_\nu(x)\}.$$

We finally get

$$I_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos hx,\quad I_{1/2}=\sqrt{\frac{2}{\pi x}}\sin hx,$$

$$K_{1/2}(x)=\frac{\pi}{2}\sqrt{\frac{2}{\pi x}}e^{-x}=\sqrt{\frac{\pi}{2x}}e^{-x}, \tag{14}$$

$$A_0(t)=\Gamma\sqrt{\frac{t}{\pi}\frac{e^{-iE_0t}}{\sqrt{2\Gamma}}}K_{1/2}(\Gamma t)=\frac{1}{2}e^{-iE_0t}e^{-\Gamma t}.$$

So we conclude by use of the method of steepest descent that just by retaining the leading term in the expansion of the direct integral from zero to infinity one obtains a good approximation to the decay amplitude. This result in fact, corresponds to the traditional expression for the decay amplitude; this means that the contribution of the integration from minus infinity to zero is generally negligible.

[a]On leave from Bar-Ilan University.
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Asymptotic form of the penetration probability of the quantum harmonic oscillator into the classically forbidden region

D. E. Atems and J. M. Wadehra
Department of Physics and Astronomy, Wayne State University, Detroit, Michigan 48202

(Received 18 July 1994; accepted 18 October 1994)

The penetration probability of the quantum harmonic oscillator into the classically forbidden region is examined in the limit of large quantum numbers using analytical methods. An expression is derived for the asymptotic form of the penetration probability, and is shown to be in good agreement with an earlier result obtained by numerical methods. The asymptotic form of the probability density in the region between the classical turning points is also presented and found to have a simple physical interpretation. © *1995 American Association of Physics Teachers.*

The harmonic oscillator is one of the most ubiquitous and useful idealizations in physics and physical chemistry, and both its classical and quantum versions have been studied extensively. Surprisingly, however, little seems to be known about the quantum mechanical probability $P(n)$ of finding the oscillator outside the classical turning points when the quantum number n is large, notwithstanding the obvious expectation, based on the correspondence principle, that this probability must somehow tend to zero as the quantum number tends to infinity. A recent article[1] presented a conjecture, supported by a purely *numerical* investigation, that this probability has the asymptotic form

$$P(n)=A(n+1/2)^{-1/3}-B(n+1/2)^{-1}+\cdots \tag{1}$$

as $n\to\infty$, with $A\approx 0.133\,970$ and $B\approx 0.011\,907$. The main purpose of the present article is to derive an asymptotic formula for this penetration probability using an *analytic* argument based on the behavior of the harmonic oscillator eigenfunctions in the vicinity of the classical turning points for large quantum numbers. It will be shown that this result is identical in form to Eq. (1). Closed-form expressions will be derived for the coefficients A and B, and their values will be found to be quite close to those quoted above. The derivation will also yield the order of the first neglected term; this result will be discussed with the help of a numerical analysis. As a further demonstration of the usefulness of the asymptotic forms of the harmonic oscillator eigenfunctions, a formula for the quantum probability density *inside* the classical turning points in the large-n limit will be briefly derived and seen to have a simple physical interpretation.

The energy eigenstate of the quantum harmonic oscillator corresponding to the quantum number n has energy $E_n=(n+1/2)\hbar\omega$ relative to the potential minimum, and the corresponding normalized eigenfunction is given by

$$\psi_n(x)=(2^n n!\sqrt{\pi}x_0)^{-1/2}e^{-x^2/2x_0^2}H_n(x/x_0),$$

where the H_n are the Hermite polynomials, and $x_0=\sqrt{\hbar/m\omega}$. The classical turning points occur at

$$x=\pm\sqrt{2\nu}x_0,$$

where for convenience, both here and later, we put $\nu\equiv n+1/2$. The probability of finding the oscillator outside the classical turning points is therefore

$$P(n)=2\int_{\sqrt{2\nu}x_0}^{\infty}dx|\psi_n(x)|^2$$

$$=2(2^n n!\sqrt{\pi}x_0)^{-1}\int_{\sqrt{2\nu}x_0}^{\infty}dx\; e^{-x^2/x_0^2}[H_n(x/x_0)]^2$$

or, changing the variable of integration from x to $y=x/x_0$,

$$P(n)=\frac{1}{2^n n!}\frac{2}{\sqrt{\pi}}\int_{\sqrt{2\nu}}^{\infty}dy\; e^{-y^2}[H_n(y)]^2. \tag{2}$$

