Exam III solutions

For an arbitrary direction \((\theta, \phi)\) we get eigen-spinors using Eq. (4.155) as

\[
\chi_+ = \begin{bmatrix} \cos(\theta/2) \\ i\phi \\ e^{i\phi} \sin(\theta/2) \end{bmatrix}, \quad \chi_- = \begin{bmatrix} -i\phi \\ \sin(\theta/2) \end{bmatrix}
\]

For \(x\) axis, \(\theta = \pi/2, \phi = 0\)

\[
\implies \chi^{(x)}_\pm = \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \frac{1}{\sqrt{2}}
\]

So first measurement gives \(\pm h/2\) with unit probability. Spinor collapses to \(\chi_{\pm}^{(x)}\).

For \(y\) axis, \(\theta = \pi/2, \phi = \pi/2\)

\[
\implies \chi_{\pm}^{(y)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \chi_{\pm}^{(y)} = \frac{-i}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Probability of getting \(\pm h/2\) on second measurement is

\[
|\chi_{\pm}^{(y)} \chi_{\pm}^{(x)}|^2 = \left| \frac{1 + i}{2} \right|^2 = \frac{1}{2}
\]

Now spinor collapses to \(\chi_{\pm}^{(y)}\).
Case (i) Collapse to $X^+_{+}$

$P(\text{measuring } \hbar/2 \text{ on third measurement}) = \frac{1}{\sqrt{2}}^2 = \frac{1}{2}$

Same logic for $-\hbar/2$, $P = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Case (ii) Collapse to $X^-_{-}$

$P_\pm = \frac{1}{2}$ again.

Combine all cases,

$P(S_x = \hbar/2) = 1$, $P(S_x = -\hbar/2) = 0$

$P(S_y = \pm \hbar/2) = \frac{1}{2}$

$P(S_z = \pm \hbar/2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
2) Notation is $P \equiv L=1$, $D \equiv L=2$, $F \equiv L=3$,

$$\frac{2S+1}{L+J}, \text{where } J = |L-S|, |L-S|+1, \ldots, L+S$$

(i) $^3P_0 \equiv L=1, S=1, J = |L-S| = 0$

(ii) $^5D_3 \equiv L=2, S=2, J = 3 = L+S-1 \geq |L-S| = 0$

(iii) $^3F_1 \equiv L=3, S=1, J=1 < |L-S|=2$

$^3F_1$ this case cannot be valid.

3 (a) Degeneracy is $2S+1 = 4, S = 3/2$.

so 4 electrons can have same spatial wavefunctions.

(b) $\frac{g}{2S+1} = \frac{8}{4} = 2$, $\Rightarrow$ filled bands

$\Rightarrow$ insulator or semiconductor.

(c) $\frac{g}{2S+1} = \frac{13}{4} = 3.25$, $\Rightarrow$ unfilled bands

$\Rightarrow$ conductor.