Examination II for PHYS 4310/5310, Spring 2009

1. Consider the observables A = x and B = L_z.
(a) Write a simple expression for the lower bound of the product of uncertainties for $\sigma_A\sigma_B$. (**1 point**)
(b) Simplify the answer in (a) as much as you can. (**1 point**)
(c) Evaluate the product $\sigma_A\sigma_B$ in the hydrogen atom state $\psi_{n,\ell,m}$. (**1 point**)
(d) What can you say about $\langle y \rangle$ in this state? (**1 point**)

2. A particle of mass m lies in a spherically symmetric potential given by V(r) = $(kr^2)/2$, where k is positive constant of appropriate dimensions and r is the distance of the particle from the origin. All relevant numerical parameters in your answer should only involve universal constants and k.
(a) Consider a solution that may be derived in analogy to the solution of problem 4.2 of the text. Using the technique write down two or more simple linear differential equations to solve the time independent Schrödinger equation for this potential. You need not derive these equations. (**3 points**)
(b) Directly write down the solutions of these equations. You need not derive the solutions. (**2 points**)
(c) Write down expressions for the eigen-value energies for the problem. (**1 point**)
(d) What is the expectation value of $r^2\sin^2(\theta)\sin^2(\phi)$, where θ and ϕ are polar and azimuthal angles of a spherical polar coordinate system? (**1 point**)

3. Two particles each of mass m are fixed at the two ends of a massless rigid rod of length b. The center of the rod is held fixed, though the rod is free to rotate in three dimensions. Ignore gravitational forces.
(a) Write expressions for the classical Hamiltonian H of the system and also the total angular momentum L of the system about the center of the rod, in terms of m, b and relevant physical dynamical variables. (**1 point**)
(b) Combine results in part (a) to write an expression for H in terms of L. (**1 point**)
(c) Solve for the energy eigen-functions and eigen-values of the quantum Hamiltonian corresponding to the one in part (b). (**3 points**)
(d) Quantify the degeneracy of each eigen-function found in part (c). (**1 point**)

4. An electron in a hydrogen atom was measured to have a total energy E = $E_1/25$ where E_1 is its ground state energy. The z component of its orbital angular momentum was measured to be $(3h)/(2\pi)$, where h is Planck's constant.
(a) Write a general expression for its wavefunction, in terms of the energy eigen-functions of the problem. (**1 point**)
(b) Find the expectation value of L^2 for this wavefunction. (**2 points**).

1] (a) $[A,B] = [x, L_z] = xL_z - L_z x = x\{xp_y - yp_x\}$
$- \{xp_y - yp_x\}x = -i\hbar y$

$\Rightarrow$ from Eq. 3.62 of the text that

$$\sigma_A^2 \sigma_B^2 \geq \left[\frac{\langle[A,B]\rangle}{2i}\right]^2 = \frac{\hbar^2}{4}\langle y\rangle$$

(b) $\Rightarrow \sigma_A \sigma_B \geq \frac{\hbar}{2}|\langle y\rangle|$

(c) $\sigma_B = 0$ in ψ_{nlm}, $\because L_z\psi_{nlm} = m\hbar\psi_{nlm}$

(d) $\sigma_A\sigma_B = 0 \Rightarrow 0 \geq \frac{\hbar}{2}|\langle y\rangle| \Rightarrow \langle y\rangle = 0$.

2] $V(r) = \frac{k}{2}r^2 = \frac{k}{2}(x^2+y^2+z^2)$

(a) In analogy we get three equations

$$-\frac{\hbar^2}{2m}\frac{d^2}{d\begin{bmatrix}x\\y\\z\end{bmatrix}^2}\begin{bmatrix}\psi_x\\ \psi_y\\ \psi_z\end{bmatrix} + \frac{k}{2}\begin{bmatrix}x\\y\\z\end{bmatrix}^2\psi_{x,y,z} = E_{x,y,z}\,\psi_{x,y,z}$$

where $\psi(x,y,z) = \psi_x\psi_y\psi_z$, $E = E_x + E_y + E_z$.

(b) Each of these is a simple harmonic oscillator, So solutions are of the form of Eqs. 2.85 and 2.83.

$$E_{x,y,z} = \left(n_{x,y,z} + \frac{1}{2}\right)\hbar\omega, \text{ where } \omega = \sqrt{k/m}$$

$n_{x,y,z} = 0, 1, 2, \ldots$

can be varied independent of each other.

Hence $E = \left[n_x + n_y + n_z + \frac{3}{2}\right]\hbar\omega$

$$\psi_m(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{(2^m)(m!)}} H_m(\xi) e^{-\xi^2/2}$$

where $H_m(\xi)$ is the m^{TH} order Hermite polynomial given by Eq. 2.86 of the text,

$$H_m(\xi) = (-1)^m e^{\xi^2} \left(\frac{d}{d\xi}\right)^m e^{-\xi^2}$$

$$q = x, y, z, \quad \xi \equiv q\sqrt{\frac{m\omega}{\hbar}}.$$

(d) $r^2 \sin^2\theta \sin^2\phi = y^2$; $\langle y^2 \rangle = \iiint \psi_x^* \psi_y^* \psi_z^* y^2 \psi_x \psi_y \psi_z \, dx\,dy\,dz$

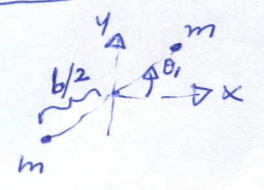
$$= \int |\psi_y|^2 y^2 dy = \int \psi_y^* y^2 \psi_y dy = \frac{\langle \frac{1}{2} k y^2 \rangle}{(k/2)}$$

$$= \frac{(n_y + \frac{1}{2})\frac{\hbar\omega}{2}}{(k/2)} = \left(n_y + \frac{1}{2}\right)\frac{\hbar\omega}{k} = \left(n_y + \frac{1}{2}\right)\left[\hbar/\sqrt{mk}\right]$$

3] $H = T + V$, $V = 0$, $T = \frac{m}{2}\left[v_1^2 + v_2^2\right] = \frac{m}{2} b^2 \left[\dot\theta_1^2 + \dot\theta_2^2\right] = m b^2 \dot\theta_1^2$

$\bar{L} = \cancel{\frac{m\bar{b}}{2}[\bar{v}_1 + \bar{v}_2]}$, $L^2 = m^2 b^2 v^2$

$= m b v \hat{k}$



Figure

$$H = m b^2 \dot\theta_1^2 = m b^2 \frac{v^2}{b^2} = m v^2 = m \frac{L^2}{m^2 b^2}$$

$\Rightarrow H = \frac{L^2}{m b^2} \Rightarrow H\psi = E\psi$ gives $L^2 \psi = m E b^2 \psi$

$\Rightarrow \psi = Y_\ell^m(\theta, \phi) \Rightarrow m E b^2 = \ell(\ell+1)\hbar^2$

$$\Rightarrow E_\ell = \frac{\ell(\ell+1)\hbar^2}{mb^2}, \quad \ell = 0, 1, 2, \ldots$$

Degeneracy $= 2\ell+1$.

4] (a) Given $E = E_1/25 = E_1/5^2 \Rightarrow n = 5$

$L_z = 3\hbar \Rightarrow m = 3$

$$\Rightarrow \psi = \sum_{\ell=3}^{4} c_\ell \psi_{5,\ell,3}(\vec{r}) . = c_4 \psi_{5,4,3} + c_3 \psi_{5,3,3}$$

Note we need $|c_4|^2 + |c_3|^2 = 1$ for normalization

(b) $\langle L^2 \rangle = \hbar^2 \left[|c_4|^2 (20) + |c_3|^2 (12) \right]$.