Examination I for PHYS 4310/5310, Spring 2009

1. A free particle of mass m is in an initial state ψ (x, 0). The functional form of this initial state is identical to the first excited energy eigen-state of an infinite square well potenmtial of well-width b.
(a) Write an expression for ψ (x, 0). (**1** point)
(b) This state may be expanded for all future time (t > 0), in the form of energy eigenfunctions of a free particle such that

$$\psi(x,t) = \int_{-\infty}^{\infty} c(p)\left[\exp\left(\frac{ipx}{\hbar}\right)\right]\left[\exp(if(p,t))\right]dp.$$

Write an expression for the function f(p,t) in terms of its arguments and other known constants of the problem. (**1** point)
(c) Using the t = 0 form of this state find c(p). (**2** points)

2. A particle in one dimension is in the ground state, of a potential
V(x) = -α$\delta(x-a)$, where a and α are positive constants of appropriate dimensions.
(a) Write (you need not derive) the ground state wavefunction ψ (x, t) of the particle. (**1** point)
(b) Find the expectation value of the Hamiltonian in this state. Hence, find the expectation value of the kinetic energy in this state. (**3** points)

3. A particle of mass m lies in an infinite square well potential, with walls at x = 0 and x = b. It has an initial wavefunction $\psi(x,0) = A[3\psi_4(x) + 4\psi_3(x)]$. The energy eigenfunctions for this potential are denoted by $\psi_n(x)$, n = 1, 2, …. Work out all parts only at t = 0. Express all answers in terms of numerical constants, m, b, and $\hbar$.
(a) Find A. (**1** point)
(b) If a measurement is conducted for the particle energy what values may be obtained? (**1** point)
(c) If the energy is measured twice in succession, derive the probability of obtaining an energy $(8/m)((\pi\hbar)/b)^2$, on the second measurement. Assume that you do not know the outcome of the first measurement. (**2** points)
(d) After this second measurement, where an energy $(8/m)((\pi\hbar)/b)^2$ was obtained, the position of the particle was measured. What would be the probability of getting a measurement in the range $x_0 \pm$ (dx/2)? (**1** point)

4. Consider a potential V(x) which is infinite for x < 0. It takes a negative value $-V_0$, when 0 < x < b and vanishes for x > b.
(a) Solve for the most general solution of the time independent Schrödinger's equation in different regions of space for the case $-V_0 < E < 0$. (**2** points)

(b) Define appropriate integration constants and eliminate solutions which will lead to unphysical results. (**2** points)
(c) Derive the quantization condition for E if such a condition exists. If it does not exist explain why this is so. (**2** points)
(d) State how you would normalize the energy eigenfunction that emerges. You need not finish the normalization exercise. (**1** point)

Examination I solutions

1] (a) $\Psi(x,0) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{2\pi x}{a}\right)$ from Eq. (2.28) of the text.

(b) Since it is a free particle $E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar p^2}{2m\hbar} = \frac{p^2}{2m}$

$$\Rightarrow f(p,t) = -\frac{Et}{\hbar} = \frac{-p^2 t}{2\hbar m}$$

(c) $$\Psi(x,0) = \int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}}\, c(p)\, dp = \hbar \int_{-\infty}^{\infty} e^{ikx}\, c(\hbar k)\, dk$$

$$\Rightarrow \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ik'x}\, dx = \hbar \int_{-\infty}^{\infty} c(\hbar k)\, dk \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx$$

$$= \hbar \int_{-\infty}^{\infty} c(\hbar k)\, dk\, (2\pi)\, \delta(k-k')$$

$$\Rightarrow c(\hbar k) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi\hbar}\left(\frac{2}{a}\right)^{1/2}\left(\frac{1}{2i}\right) \int_{-\infty}^{\infty} \left[e^{ix\left[\frac{2\pi}{a} - k\right]} - e^{-ix\left[\frac{2\pi}{a} + k\right]} \right] dx$$

$$\Rightarrow c(p) = \frac{1}{2i\hbar}\left(\frac{2}{a}\right)^{1/2} \left[\delta\left[\frac{p}{\hbar} - \frac{2\pi}{a}\right] - \delta\left[\frac{p}{\hbar} + \frac{2\pi}{a}\right] \right]$$

$$= \frac{i}{\sqrt{2a}} \left[\delta\left[p + \frac{2\pi\hbar}{a}\right] - \delta\left[p - \frac{2\pi\hbar}{a}\right] \right]$$

2] From Eq. (2.129) we get

(a) $\Psi(x,t) = \left[\sqrt{\dfrac{m\alpha}{\hbar^2}}\right] \exp.\left[\dfrac{-m\alpha|x-a|}{\hbar^2} + \dfrac{i m\alpha^2 t}{2\hbar^2}\right]$

(b) $\langle\Psi|\hat{H}|\Psi\rangle = E = \dfrac{-m\alpha^2}{2\hbar^2}$

$$\langle\hat{H}\rangle - \langle\hat{V}\rangle = \langle\hat{T}\rangle$$

$$\langle\hat{V}\rangle = |\Psi(a,t)|^2[-\alpha] = -\frac{m\alpha^2}{\hbar^2}\exp.\left[\frac{-2m\alpha(0)}{\hbar^2}\right]$$

$$= 2E$$

$\Rightarrow \langle\hat{T}\rangle = E - 2E = -E = \dfrac{m\alpha^2}{2\hbar^2} > 0$ as expected.

3] (a) $\int_0^a |\Psi(x,0)|^2 dx = 1 \Rightarrow |A|^2[9+16] = 1$

$\Rightarrow |A| = 1/5$. We used $\int_0^a \psi_m^* \psi_n \, dx = \delta_{m,n}$.

(b) Possible values of E are E_3 and E_4

where $E_n = \dfrac{\hbar^2 k_n^2}{2m} = n^2\left[\dfrac{\hbar^2\pi^2}{2ma^2}\right]$

Note $b \equiv a$.

Note $\dfrac{8\pi^2\hbar^2}{mb^2} = 4^2\left[\dfrac{\pi^2\hbar^2}{2mb^2}\right] = E_4$

(c) First measurement gives E_4 with probability $= 9/(25)$. Then for second measurement we get unit probability due to collapse of Ψ.

If first measurement gives E_3 then due to collapse of Ψ there is zero probability of measuring E_4 on the next measurement.

Hence total chance is $9/25 = 0.36$.

(d) $P\left(x_0 \pm \dfrac{dx}{2}\right) = \left(\dfrac{2}{a}\right)\sin^2\left(\dfrac{4\pi x}{a}\right) dx$

4] (a) $\psi(x) = A\sin(kx) + B\cos(kx)$, $\forall\, 0 < x < b$

$= Ce^{Kx} + De^{-Kx}$, $\forall\, x > b$

$= 0$, $\forall\, x < 0$

$K = \sqrt{\frac{-2mE}{\hbar^2}}$, $k = \sqrt{\frac{2m}{\hbar^2}(+V_0 + E)}$

(b) $D \neq 0$, $C = 0$, $\because \psi(x) \to 0$ as $x \to \infty$, for $\int |\psi|^2 dx < \infty$,

$B = 0$, $A \neq 0$, $\because \psi(0) = 0$.

(c) $\psi(b^-) = \psi(b^+)$, $\psi'(b^-) = \psi'(b^+)$

$\Rightarrow A\sin(kb) = De^{-Kb}$, $Ak\cos(kb) = -KDe^{-Kb}$

$\Rightarrow k\cot(kb) = -K$ is the quantization condition for E.

This also gives $D = \sin(kb)e^{Kb}A$

$\Rightarrow \psi(x) = A\sin(kx)$, $\forall\, 0 < x < b$

$= A\sin(kb)\,\exp.[K(b-x)]$

(d) $\int_{-\infty}^{\infty} |\psi|^2 dx = 1 \Rightarrow |A|^{-2} = \int_0^b \sin^2(kx)\,dx + \sin^2(kb)e^{2Kb}\int_b^{\infty} e^{-2Kx}\,dx.$