The Uncertainty Principle

We define the expectation value $\langle \hat{A} \rangle$ of an operator as $\boxed{3\text{-}23}$

$\langle \Psi | \hat{A} | \Psi \rangle$. It is calculated in the state $|\Psi\rangle$. ~~so~~ $|\Psi\rangle$ is not some specific state unless so stated. Note that $\langle \Psi | \hat{A} | \Psi \rangle$ is a scalar. If $\hat{A} = \hat{A}^{\dagger}$

then $\langle \Psi | \hat{A} | \Psi \rangle^{\dagger} = \langle \Psi | \hat{A} | \Psi \rangle$

$$\Rightarrow \langle \Psi | \hat{A} | \Psi \rangle \in R.$$

Define $|f\rangle \equiv [\hat{A} - \langle \hat{A} \rangle] |\Psi\rangle$

and $|g\rangle \equiv [\hat{B} - \langle \hat{B} \rangle] |\Psi\rangle$

Define $\sigma_A^2 \equiv \langle f | g \rangle$ and $\sigma_B^2 \equiv \langle g | g \rangle$

$\therefore \sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle$ which by Schwarz inequality gives

$$\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

$$\therefore \sigma_A^2 \sigma_B^2 \geq |\langle f | g \rangle|^2 \longrightarrow 2$$

$\forall z \in C$ we have $|z|^2 = [Re.(z)]^2 + [Im.(z)]^2$

$$\therefore |z|^2 = \left(\frac{z+z^*}{2}\right)^2 + \frac{(z-z^*)^2}{(2i)^2}$$

$$\therefore |z|^2 \geq \left[\frac{z-z^*}{2i}\right]^2$$

3-24

$$\therefore |\langle f|g\rangle|^2 \geq \left[\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right]^2 \longrightarrow 3$$

Now $\langle f| = \langle \Psi|(\hat{A}^{\dagger} - \langle \hat{A}^{\dagger}\rangle)$

$= \langle \Psi|(\hat{A} - \langle \hat{A}\rangle)$, $\forall \hat{A} = \hat{A}^{\dagger}$

$$\therefore \langle f|g\rangle = \langle \Psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle \hat{B}\rangle)|\Psi\rangle$$

$$= \langle \Psi|\hat{A}\hat{B}|\Psi\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$= \langle \Psi|\hat{A}\hat{B}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

~~So~~ where we used $\langle\Psi|\Psi\rangle = 1$

Similarly $\langle g|f\rangle = \langle\Psi|\hat{B}\hat{A}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$

$$\Rightarrow \langle f|g\rangle - \langle g|f\rangle = \langle\Psi|\hat{A}\hat{B} - \hat{B}\hat{A}|\Psi\rangle$$

$$= \langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle \longrightarrow 4$$

Combining 2, 3 and 4 we get

$$\sigma_A^2 \sigma_B^2 \geq \left[\frac{\langle[\hat{A},\hat{B}]\rangle}{2i}\right]^2$$

or

$$\sigma_A \sigma_B \geq \left| \frac{\langle [\hat{A}, \hat{B}] \rangle}{2i} \right|$$

We know as a special case of 3-25

$\hat{A} \equiv \hat{X}$ and $\hat{B} \equiv \hat{P}$ that

$$[\hat{X}, \hat{P}] = i\hbar \Rightarrow \langle [\hat{X}, \hat{P}] \rangle = i\hbar$$

$$\therefore \quad \sigma_x \sigma_p \geq \left(\frac{\hbar}{2}\right) \longrightarrow \text{(5)}$$

Equation (5) is the original Heisenberg's uncertainty principle.

σ_A is also denoted Δa

$\therefore$ $(\Delta x)(\Delta p) \geq \hbar/2$ is another form.

The Energy Time Uncertainty principle

Let $\hat{\varphi}$ be an observable $\Rightarrow \hat{\varphi} = \hat{\varphi}^\dagger$.

$$\therefore \frac{d}{dt}\langle \hat{\varphi} \rangle = \frac{d}{dt}\langle \Psi | \hat{\varphi} | \Psi \rangle = \langle \dot{\Psi}(t) | \hat{\varphi} | \Psi(t) \rangle + \langle \Psi | \dot{\hat{\varphi}} | \Psi \rangle + \langle \Psi | \hat{\varphi} | \dot{\Psi} \rangle$$

Now $\quad i\hbar | \dot{\Psi}(t) \rangle = \hat{H} | \Psi(t) \rangle$

$$\Rightarrow \quad | \dot{\Psi}(t) \rangle = \frac{-i\hat{H}}{\hbar} | \Psi(t) \rangle$$

$$\langle \dot{\Psi}(t) | = \frac{i}{\hbar} \langle \Psi(t) | \hat{H}$$

$$\Rightarrow \frac{d}{dt}\langle \hat{\varphi} \rangle = \left\langle \frac{\partial \hat{\varphi}}{\partial t} \right\rangle + \frac{i}{\hbar} \langle \psi | \hat{H}\hat{\varphi} - \hat{\varphi}\hat{H} | \psi \rangle$$

$$= \langle \psi | \frac{\partial \hat{\varphi}}{\partial t} | \psi \rangle + \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{\varphi}] | \psi \rangle$$

Now suppose $\frac{\partial \hat{\varphi}}{\partial t} = \hat{0} \Leftrightarrow \hat{\varphi}$ does not have an explicit time dependence

$$\Rightarrow \frac{d}{dt}\langle \hat{\varphi} \rangle = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{\varphi}] | \psi \rangle$$

But by the uncertainty principle we get

$$\sigma_H^2 \sigma_\varphi^2 \geq \left[\frac{1}{2i}\langle [\hat{H}, \hat{\varphi}] \rangle\right]^2 = \left(\frac{\hbar}{2}\right)^2 \left(\frac{d\langle \hat{\varphi} \rangle}{dt}\right)^2$$

$$\Rightarrow \frac{\sigma_H \sigma_\varphi}{\left|\frac{d\langle \hat{\varphi} \rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

Define $\Delta t \equiv \dfrac{\sigma_\varphi}{\left|\frac{d\langle \hat{\varphi} \rangle}{dt}\right|}$

$$\Delta E \equiv \sigma_H$$

$$\Rightarrow \Delta E \Delta t \geq \hbar/2$$

The Minimum Energy Wave-function.

We want $\sigma_A \sigma_B = \hbar/2$

Let $\hat{A} \equiv \hat{X}$, $\hat{B} \equiv \hat{P}$

In the derivation of the uncertainty principle we had two inequalities

The Schwarz inequality

$$\langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2$$

To make this an equality we need

$$|g\rangle = c|f\rangle$$
$$\Rightarrow \langle f|g\rangle = c\langle f|f\rangle$$

Also we used $|\langle f|g\rangle|^2 \geq (\text{Im}\,\langle f|g\rangle)^2$

To make this an equality we need

$$\text{Re}(\langle f|g\rangle) = 0 \Rightarrow \text{Re}.(c\langle f|f\rangle) = 0$$
$$\Rightarrow \text{Re}(c) = 0 \Rightarrow c = ia, \quad a \in \mathbb{R}.$$

$$\therefore \quad |g\rangle = ia|f\rangle$$

$$\Rightarrow [\hat{P} - \langle\hat{P}\rangle]|\Psi\rangle = ia(\hat{X} - \langle\hat{X}\rangle)|\Psi\rangle$$

$$\Rightarrow \left[\left(\frac{-i\hbar}{1}\right)\frac{d}{dx} - \langle\hat{P}\rangle\right]\Psi(x) = ia(x - \langle\hat{X}\rangle)\Psi(x)$$

$$\Rightarrow \Psi(x) = A e^{-\left[\frac{a(x-\langle\hat{X}\rangle)^2}{2\hbar}\right]} e^{\frac{ix\langle p\rangle}{\hbar}}.$$

This is a Gaussian