Analytic Solution to HO.

We want to solve TISE for
$V(x) = \frac{m\omega^2}{2} x^2$

$\Rightarrow$ $\hat{H}\psi_E = E\psi_E$ becomes

$$\frac{d^2\psi_E(x)}{dx^2} = \frac{2m}{\hbar^2}\left[\frac{m\omega^2}{2}x^2 - E\right]\psi_E(x) \longrightarrow (2.70)$$

Let $y = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$. Note book uses ξ for y.

$$\Rightarrow \quad \frac{d^2\psi_E(y)}{dy^2} = \left(y^2 - K\right)\psi_E(y).$$

where $K \equiv \frac{2E}{\hbar\omega}$

We now look at approximate solution at $y^2 \gg K$

$$\Rightarrow \quad \psi_E''(y) \approx y^2 \psi_E(y)$$

$$\Rightarrow \quad \psi_E(y) \approx A e^{-y^2/2} + B e^{y^2/2}$$

The term $e^{y^2/2} \to \infty$ as $y \to \infty$ and will not be normalizable. So we try a solution

$$\psi_E(y) = h(y)\, e^{-y^2/2}$$

Putting this into Eq. (2.70) gives

$$h''(y) - 2yh'(y) + (K-1)h = 0 \longrightarrow (2.78)$$

We now try a series solution

$$h(y) = a_0 + a_1 y + a_2 y^2 + \dots = \sum_{j=0}^{\infty} a_j y^j$$

$$\Rightarrow h'(y) = \sum_{j=0}^{\infty} j a_j y^{j-1} \quad \text{and} \quad h''(y) = \sum_{j=0}^{\infty} j(j-1) a_j y^{j-2}$$

$$\Rightarrow h''(y) = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} y^j$$

Using these series in Eq. (2.78) gives

$$\sum_{j=0}^{\infty} y^j \left[(j+2)(j+1) a_{j+2} - 2j a_j + (K-1) a_j \right] = 0$$

For this to hold true $\forall y \in R$

$$\Rightarrow (j+2)(j+1) a_{j+2} + (K-1-2j) a_j = 0$$

$$\Rightarrow a_{j+2} = \left[\frac{1+2j-K}{(j+2)(j+1)} \right] a_j \longrightarrow (2.81)$$

This is a recursion formula for a_j. Note that if we know a_0 and a_1 then all higher order co-efficients a_j, $j > 1$ are known. Thus $h(y)$ and hence $\psi(y)$ is completely defined by just two constants a_0 and a_1 as expected for a second order differential equation.

Now we ~~obs~~ observe that
$\forall j \gg 1$

$$a_{j+2} \approx \frac{2a_j}{j} \quad \text{from Eq. } (2.81)$$

$$\Rightarrow a_j \approx \frac{a_{j-2}}{\left(\frac{j-2}{2}\right)} \approx \frac{a_{j-4}}{\left(\frac{j-2}{2}\right)\left(\frac{j-4}{2}\right)} \cdots,$$

$$\Rightarrow a_j \approx \frac{C}{(j/2)!} \quad \Rightarrow h(y) \approx C \sum_j \frac{y^j}{\left(\frac{j}{2}\right)!}$$

$$\Rightarrow h(y) \approx C \sum_j \frac{y^{2j}}{j!} \approx Ce^{y^2}$$

$$\Rightarrow \psi(y) \approx \left(Ce^{y^2}\right) e^{-y^2/2} \approx Ce^{y^2/2}$$

$\Rightarrow \psi(y) \to \infty$ as $y \to \infty$. This is not a good solution.

$\Rightarrow$ The recursion relation Eq. (2.81) has to stop at some point.

The only way that will happen is if $\exists$ an integer $n \ni$

$$K = 2n+1, \quad n \geq 0.$$

$$\Rightarrow \frac{E_n}{\hbar\omega} = \left(n+\frac{1}{2}\right) \text{ or } E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

Note that this is the same energy spectrum we got earlier using $\hat{a}_{\pm}$.

Now $a_{n+2} = a_{n+4} = a_{n+6} \cdots = 0$

However this says nothing about $a_{n-1}, a_{n-3}, \cdots$ and $a_{n+1}, a_{n+3} \cdots$

$\Rightarrow$ if n is odd we need to choose $a_0 = 0$

and

if n is even then $a_1 = 0$.

$\Rightarrow$ $h(y) = h_n(y) = n^{TH}$ order polynomial

if n is odd $h_n(y) = -h_n(-y)$

$$\Rightarrow \psi_n(y) = h_n(y) e^{-y^2/2} = -\psi_n(-y)$$

if n is even $\psi_n(y) = h_n(y) e^{-y^2/2} = \psi_n(-y)$

$$h_n(y) = a_0 H_n(y) \quad , \forall n \text{ even}$$

$$= a_1 H_n(y) \quad , \forall n \text{ odd}$$

$H_n(y)$ are the Hermite polynomials defined by Eq. (2.81)

$$H_0(y) = 1, \; H_1(y) = 2y, \; H_2(y) = 4y^2 - 2.$$

Now we can determine a_0 and a_1 from normalization of $\psi_n(y)$ so we ~~gen~~ get

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{H_n(y)}{\sqrt{2^n (n!)}} e^{-y^2/2}, \text{ where } y = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$$