Bound & Scattering States

$E < V(\pm\infty) \Rightarrow$ bound state $\Psi_E(x,t)$

$$\Rightarrow \int_{-\infty}^{\infty} |\Psi_E(x,t)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} \Psi_{E_m}^* \Psi_{E_n} dx \propto \delta_{n,m}$$

E values are discrete

$E > V(\pm\infty) \Rightarrow$ scattering state $\Psi_E(x,t)$

$$\Rightarrow \int_{-\infty}^{\infty} |\Psi_E(x,t)|^2 dx \text{ is infinite}$$

or $$\int_{-\infty}^{\infty} \Psi_{E'}^* \Psi_E \, dx \propto \delta(E-E').$$

Note :→ Classical bound states need not be quantum bound states!

See Fig. on next page, Fig 2.12 of the text.

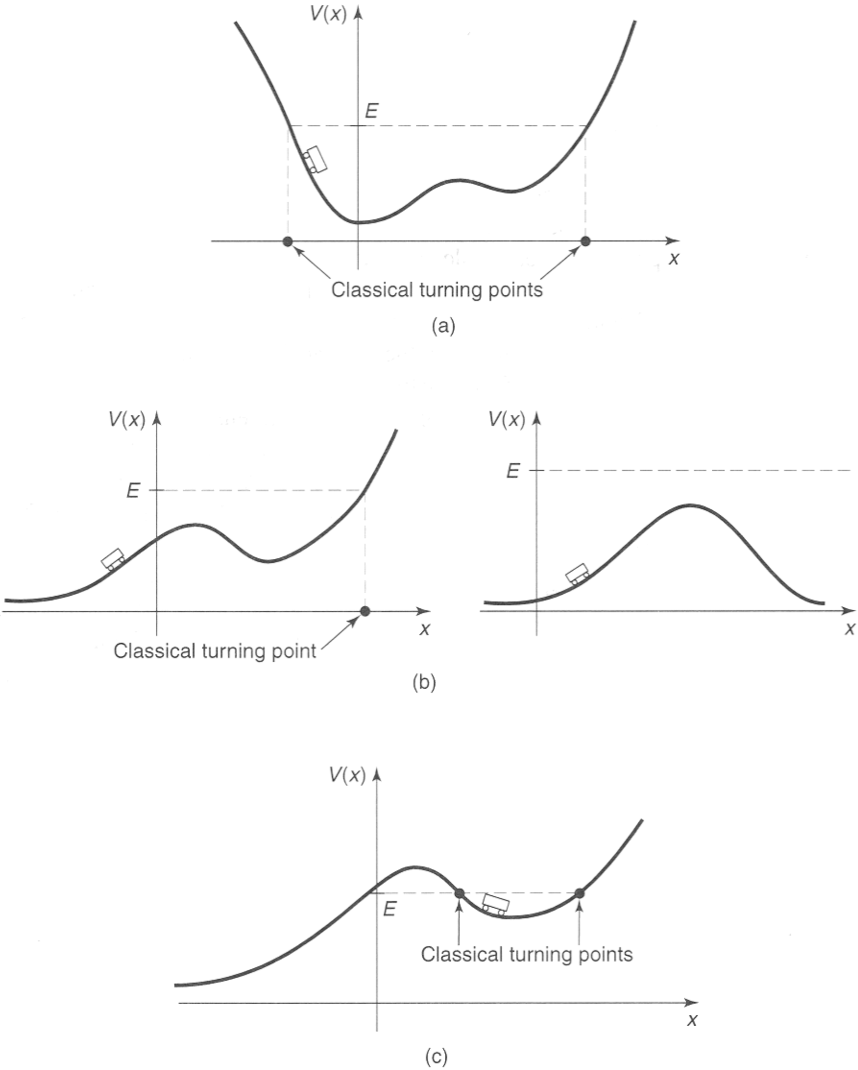

FIGURE 2.12: (a) A bound state. (b) Scattering states. (c) A *classical* bound state, but a quantum scattering state.

everywhere, it only allows scattering states.[34] In this section (and the following one) we shall explore potentials that give rise to both kinds of states.

[34]If you are irritatingly observant, you may have noticed that the general theorem requiring $E > V_{\text{min}}$ (Problem 2.2) does not really apply to scattering states, since they are not normalizable anyway. If this bothers you, try solving the Schrödinger equation with $E \leq 0$, for the free particle, and

The Delta Function Potential

Let $V(x) = -\alpha\delta(x)$, $\alpha > 0$.

$$\therefore \quad \hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x) \quad , \quad \forall x \in R.$$

Note $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$, $\forall x \neq 0$

We want to solve TISE $\hat{H}\psi_E = E\psi_E$

Case (i) $E < 0$

$$\Rightarrow \quad \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) = E\psi_E(x) \quad , \quad \forall x \neq 0$$

$$\Rightarrow \quad \psi_E''(x) = k^2\psi_E(x) \text{ where } k \equiv \sqrt{\frac{-2mE}{\hbar^2}}$$

Note $E < 0 \Rightarrow k$ is real and positive.

The general solution is

$$\psi_E(x) = Ae^{-kx} + Be^{kx} \quad , \quad \forall x < 0$$

$$= Fe^{-kx} + Ge^{kx} \quad , \quad \forall x > 0$$

$$\lim_{x \to \pm\infty} \psi(x) = 0 \Rightarrow A = G = 0$$

$\psi(x)$ is continuous everywhere

$$\Rightarrow \quad \psi_E(x)\Big|_{x\to 0+} = \psi_E(x)\Big|_{x\to 0-}$$

$$\Rightarrow \quad G = B$$

$$\Rightarrow \quad \psi_E(x) = B\, e^{-k|x|}, \quad \forall x \neq 0.$$

We have not found k and $\psi_E(0)$.
For this we integrate $\hat{H}\psi_E(x) = E\psi_E(x)$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon} \frac{d^2\psi_E(x)}{dx^2}\,dx + \int_{-\epsilon}^{\epsilon} V(x)\psi(x)\,dx = \underbrace{E\int_{-\epsilon}^{\epsilon}\psi(x)\,dx}_{=0}$$

because $\lim\limits_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\psi(x)\,dx = 0$

$$\Rightarrow \quad \frac{d\psi_E(x)}{dx}\Big|_{\epsilon} - \frac{d\psi_E(x)}{dx}\Big|_{-\epsilon} = \frac{-2m\alpha}{\hbar^2}\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx$$

$$= -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$\Rightarrow \quad -2Bk = -\frac{2m\alpha}{\hbar^2}\psi(0) \quad \Rightarrow \quad \psi(0) = B$$

$$k = \frac{m\alpha}{\hbar^2} \quad \Rightarrow \quad E = -\frac{\hbar^2 k^2}{2m} = \frac{-m\alpha^2}{2\hbar^2}$$

Normalization gives $B = \sqrt{k} = \sqrt{\dfrac{m\alpha}{\hbar^2}}$

$$\Rightarrow \quad \psi_E(x,t) = \left(\frac{m\alpha}{\hbar^2}\right)^{1/2} \exp.\left[-\frac{m\alpha|x|}{\hbar^2} + \frac{i m\alpha^2 t}{2\hbar}\right], \quad E<0.$$

Note that such a bound state will not exist for a δ-function barrier i.e. $\alpha > 0$.

Now let us consider unbounded states, i.e. $E > V(\pm\infty) = 0 \Rightarrow E > 0$.

Now TISE becomes $[\forall x \neq 0]$

$$\psi''(x) = -k^2\psi(x) \quad \text{where}$$

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}}, \quad E > 0$$

$$\Rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}, \quad \forall x < 0$$

$$= Fe^{ikx} + Ge^{-ikx}, \quad \forall x > 0$$

Continuity of $\psi(x)$ at $x = 0$

$$\Rightarrow \quad F + G = A + B \longrightarrow 2.133$$

$$\left.\frac{d\psi(x)}{dx}\right|_{x \to 0+} = ik(F - G)$$

$$\left.\frac{d\psi(x)}{dx}\right|_{x \to 0-} = ik(A - B)$$

Also $\psi(0) = A + B = F + G$

Now we use

$$\left.\frac{d\Psi}{dx}\right|_{0+} - \left.\frac{d\Psi}{dx}\right|_{0-} = \frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} V(x)\,\Psi(x)\,dx$$

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$$= \frac{-2m\alpha}{\hbar^2}\Psi(0)$$

$$\Rightarrow \quad ik(F-G-A+B) = \frac{-2m\alpha}{\hbar^2}(A+B)$$

$$\Rightarrow \left.\begin{aligned} F-G &= A(1+2i\beta) - B(1-2i\beta) \\ \text{where } \beta &\equiv \frac{m\alpha}{\hbar^2 k} \end{aligned}\right\} \rightarrow 2.135$$

We remember that

$$\Psi(x,t) = \left[Ae^{ikx} + Be^{-ikx}\right]e^{-\frac{iEt}{\hbar}}, \quad \forall x<0$$

The first term is a wave travelling to the right while the second term is a wave travelling to the left on the x-axis.

Now consider a case when $G=0$ initially

$$\Rightarrow B = \left(\frac{i\beta}{1-i\beta}\right)A \quad \text{and}$$

$$F = \left(\frac{1}{1-i\beta}\right)A$$

Now $|\Psi(x,t)|^2 = |F|^2 \, , \; \forall x > 0$

The $|\Psi(x,t)|_R^2 = |A|^2 \, , \; \forall x < 0$ and

we consider only the right (R) going part of $\Psi(x,t)$

Similarly $|\Psi(x,t)|_L^2 = |B|^2 \, , \; \forall x < 0$

for the left going part.

$\therefore$ The reflection co-efficient

$$R = \left. \frac{|\Psi(x,t)|_L^2}{|\Psi(x,t)|_R^2} \right|_{x<0} = \frac{\beta^2}{1+\beta^2} = \left[\frac{1}{1+\left(\frac{2\hbar^2 E}{m\alpha^2}\right)} \right]$$

$$T = \frac{\left[|\Psi(x,t)|_R^2\right]_{x>0}}{\left[|\Psi(x,t)|_R^2\right]_{x<0}} = \frac{1}{1+\beta^2} = \left[\frac{1}{1+\left(\frac{m\alpha^2}{2\hbar^2 E}\right)} \right]$$

As expected $R + T = 1$