So far we considered finite dimensional spaces where $n \in N$ was a natural number i.e. a positive integer.

Now we will consider spaces of infinite dimension. A particular such space is the space of states or Hilbert space. All vectors or states belong to the set $\mathcal{H}$.

Hilbert space is the space of states $\ni$

$$\langle \Psi | \Psi \rangle \in R$$
$$\langle \Psi | \Psi \rangle < \infty \quad \text{i.e. is finite.}$$

In general $|\Psi\rangle$ will be an infinite dimensional vector. There will be some cases like spin states when $|\Psi\rangle$ will be a finite dimensional vector or state. One particular basis set in this space is the position basis set $\{|x\rangle, x \in R\}$.

It has the properties:

① $\langle x | x' \rangle = 0 \quad , \forall x \neq x'$

② $\lim_{x' \to x} \langle x | x' \rangle \longrightarrow \infty$

③ $\displaystyle\int_{x-\epsilon_1}^{x+\epsilon_2} \langle x | x' \rangle \, dx' = 1 \quad , \forall \epsilon_1, \epsilon_2 > 0.$

Such a function is called a Dirac

delta function and is denoted by

$$\delta(x-x') = \delta(x'-x).$$

$$\therefore \int_{x-\epsilon_1}^{x+\epsilon_1} \delta(x-x')\, f(x')\, dx' = f(x).$$

$$\because \delta(x-x') \neq 0 \text{ only when } x = x'.$$

Consider $\delta'(x-x') \equiv \frac{d}{dx}\delta(x-x') \equiv -\frac{d}{dx'}\delta(x-x')$

$$\therefore \int_{x-\epsilon_1}^{x+\epsilon_2} \delta'(x-x')\, f(x')\, dx' = \frac{d}{dx}\int_{x-\epsilon_1}^{x+\epsilon_2} \delta(x-x')\, f(x')\, dx'$$

$$= \frac{d}{dx} f(x) \equiv f'(x).$$

The identity operator in this ~~pa~~ basis is

$$\hat{I} = \sum_x |x\rangle\langle x|$$

Note x is a continuous variable so we need to write

$$\hat{I} = \int \left[\,|x\rangle\langle x|\,\right] dx$$

Now $|\psi\rangle = \hat{I}|\psi\rangle$

$$= \int |x\rangle\langle x|\psi\rangle\, dx$$

We will denote the projection of $|\psi\rangle$

on to the basis $|x\rangle$, $\langle x|\Psi\rangle$ by the function $\Psi(x)$

$$\therefore \Psi(x) \equiv \langle x|\Psi\rangle, \quad \Psi^*(x) = \langle \Psi|x\rangle$$

In general $\Psi(x)$ is a complex function.
Now

$$\langle \Psi|\Psi\rangle \text{ is finite}$$

$$\Rightarrow \langle \Psi|\hat{I}|\Psi\rangle \text{ is finite}$$

$$\Rightarrow \int dx \, \langle \Psi|x\rangle\langle x|\Psi\rangle < \infty$$

$$\Rightarrow \int \Psi^*(x)\,\Psi(x)\,dx < \infty$$

$$\Rightarrow \int |\Psi(x)|^2\,dx < \infty .$$

So $\mathcal{H}$ consists of square integrable functions $\Psi(x)$.

$$\therefore |\Psi\rangle = \int |x\rangle\langle x|\Psi\rangle\,dx$$

$$= \int \Psi(x)\,|x\rangle\,dx$$

$$\Rightarrow \langle x'|\Psi\rangle = \int \Psi(x)\,\langle x'|x\rangle\,dx$$

$$= \int \Psi(x)\,\delta(x'-x)\,dx = \Psi(x')$$

$\Rightarrow$ $\Psi(x)$ is the projection of $|\Psi\rangle$ in the position basis $\{|x\rangle, x \in R\}$.

Let us begin by getting acquainted with an infinite-dimensional vector. Consider a function defined in some interval, say, $a \le x \le b$. A concrete example is provided by the displacement $f(x, t)$ of a string clamped at $x = 0$ and $x = L$ (Fig. 1.6).

Suppose we want to communicate to a person on the moon the string's displacement $f(x)$, at some time t. One simple way is to divide the interval 0–L into 20 equal parts, measure the displacement $f(x_i)$ at the 19 points $x = L/20, 2L/20, \ldots, 19L/20$, and transmit the 19 values on the wireless. Given these $f(x_i)$, our friend on the moon will be able to reconstruct the approximate picture of the string shown in Fig. 1.7.

If we wish to be more accurate, we can specify the values of $f(x)$ at a larger number of points. Let us denote by $f_n(x)$ the discrete approximation to $f(x)$ that coincides with it at n points and vanishes in between. Let us now interpret the ordered n-tuple $\{f_n(x_1), f_n(x_2), \ldots, f_n(x_n)\}$ as components of a ket $|f_n\rangle$ in a vector space $\mathbb{V}^n(R)$:

$$|f_n\rangle \leftrightarrow \begin{bmatrix} f_n(x_1) \\ f_n(x_2) \\ \vdots \\ f_n(x_n) \end{bmatrix} \tag{1.10.1}$$

The basis vectors in this space are

$$|x_i\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th place} \tag{1.10.2}$$

corresponding to the discrete function which is unity at $x = x_i$ and zero

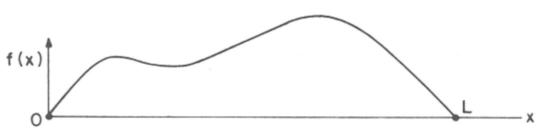

Fig. 1.6. The string is clamped at $x = 0$ and $x = L$. It is free to oscillate in the plane of the paper.

Fig. 1.7. The string as reconstructed by the person on the moon.

elsewhere. The basis vectors satisfy

$$\langle x_i \mid x_j \rangle = \delta_{ij} \qquad \text{(orthogonality)} \tag{1.10.3}$$

$$\sum_{i=1}^{n} \mid x_i \rangle \langle x_i \mid = I \qquad \text{(completeness)} \tag{1.10.4}$$

Try to imagine a space containing n mutually perpendicular axes, one for each point x_i. Along each axis is a unit vector $\mid x_i \rangle$. The function $f_n(x)$ is represented by a vector whose projection along the ith direction is $f_n(x_i)$:

$$|f_n\rangle = \sum_{i=1}^{n} f_n(x_i) \mid x_i \rangle \tag{1.10.5}$$

To every possible discrete approximation $g_n(x)$, $h_n(x)$, etc., there is a corresponding ket $\mid g_n \rangle$, $\mid h_n \rangle$, etc., and vice versa. You should convince yourself that if we define vector addition as the addition of the components, and scalar multiplication as the multiplication of each component by the scalar, then the set of all kets representing discrete functions that vanish at $x = 0$, L and that are specified at n points in between, forms a vector space.

We next define the inner product in this space:

$$\langle f_n \mid g_n \rangle = \sum_{i=1}^{n} f_n(x_i) g_n(x_i) \tag{1.10.6}$$

Two functions $f_n(x)$ and $g_n(x)$ will be said to be orthogonal if $\langle f_n \mid g_n \rangle = 0$.

Let us now forget the man on the moon and consider the maximal specification of the string's displacement, by giving its value at every point in the interval 0–L. In this case $f_\infty(x) \equiv f(x)$ is specified by an ordered infinity of numbers: an $f(x)$ for each point x. Each function is now represented by a ket $|f_\infty\rangle$ in an infinite-dimensional vector space and vice versa. Vector addition and scalar multiplication are defined just as before. Consider, however, the inner product. For finite n it was defined as

$$\langle f_n \mid g_n \rangle = \sum_{i=1}^{n} f_n(x_i) g_n(x_i)$$

in particular

$$\langle f_n | f_n \rangle = \sum_{i=1}^{n} [f_n(x_i)]^2$$

If we now let n go to infinity, so does the sum, for practically any function. What we need is the redefinition of the inner product for finite n in such a way that as n tends to infinity, a smooth limit obtains. The natural choice is of course

$$\langle f_n | g_n \rangle = \sum_{i=1}^{n} f_n(x_i) g_n(x_i) \Delta, \qquad \Delta = L/(n+1) \tag{1.10.6'}$$

If we now let n go to infinity, we get, by the usual definition of the integral,

$$\langle f | g \rangle = \int_0^L f(x) g(x)\, dx \tag{1.10.7}$$

$$\langle f | f \rangle = \int_0^L f^2(x)\, dx \tag{1.10.8}$$

If we wish to go beyond the instance of the string and consider complex functions of x as well, in some interval $a \le x \le b$, the only modification we need is in the inner product:

$$\langle f | g \rangle = \int_a^b f^*(x) g(x)\, dx \tag{1.10.9}$$

What are the basis vectors in this space and how are they normalized? We know that each point x gets a basis vector $|x\rangle$. The orthogonality of two different axes requires that

$$\langle x | x' \rangle = 0, \qquad x \neq x' \tag{1.10.10}$$

What if $x = x'$? Should we require, as in the finite-dimensional case, $\langle x | x \rangle = 1$? The answer is no, and the best way to see it is to deduce the correct normalization. We start with the natural generalization of the completeness relation Eq. (1.10.4) to the case where the kets are labeled by a continuous index x':

$$\int_a^b | x' \rangle \langle x' |\, dx' = I \tag{1.10.11}$$

where, as always, the identity is required to leave each ket unchanged. Dotting both sides of Eq. (1.10.11) with some arbitrary ket $|f\rangle$ from the right and the basis bra $\langle x|$ from the left,

$$\int_a^b \langle x | x' \rangle \langle x' | f \rangle \, dx' = \langle x | \mathrm{I} | f \rangle = \langle x | f \rangle \tag{1.10.12}$$

Now, $\langle x | f \rangle$, the projection of $|f\rangle$ along the basis ket $|x\rangle$, is just $f(x)$. Likewise $\langle x' | f \rangle = f(x')$. Let the inner product $\langle x | x' \rangle$ be some unknown function $\delta(x, x')$. Since $\delta(x, x')$ vanishes if $x \neq x'$ we can restrict the integral to an infinitesimal region near $x' = x$ in Eq. (1.10.2):

$$\int_{x-\varepsilon}^{x+\varepsilon} \delta(x, x') f(x') \, dx' = f(x) \tag{1.10.13}$$

In this infinitesimal region, $f(x')$ (for any reasonably smooth f) can be approximated by its value at $x' = x$, and pulled out of the integral:

$$f(x) \int_{x-\varepsilon}^{x+\varepsilon} \delta(x, x') \, dx' = f(x) \tag{1.10.14}$$

so that

$$\int_{x-\varepsilon}^{x+\varepsilon} \delta(x, x') \, dx' = 1 \tag{1.10.15}$$

Clearly $\delta(x, x')$ cannot be finite at $x' = x$, for then its integral over an infinitesimal region would also be infinitesimal. In fact $\delta(x, x')$ should be infinite in such a way that its integral is unity. Since $\delta(x, x')$ depends only on the difference $x - x'$, let us write it as $\delta(x - x')$. The "function," $\delta(x - x')$, with the properties

$$\begin{aligned} \delta(x - x') &= 0, \qquad x \neq x' \\ \int_a^b \delta(x - x') \, dx' &= 1, \qquad a \leq x \leq b \end{aligned} \tag{1.10.16}$$

is called the *Dirac delta function* and fixes the normalization of the basis vectors:

$$\langle x | x' \rangle = \delta(x - x') \tag{1.10.17}$$

It will be needed any time the basis kets are labeled by a continuous index such as x. Note that it is defined only in the context of an integration: the integral of the delta function $\delta(x - x')$ with any smooth function $f(x')$ is $f(x)$. One sometimes calls the delta function the sampling function, since it samples the value of the function $f(x')$ at one point‡:

$$\int \delta(x - x') f(x') \, dx' = f(x) \tag{1.10.18}$$

‡ We will often omit the limits of integration if they are unimportant.

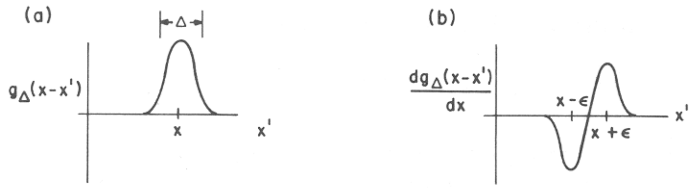

Fig. 1.8. (a) The Gaussian g_Δ approaches the delta function as $\Delta \to 0$. (b) Its derivative $(dg/dx)(x - x')$ approaches $\delta'(x - x')$ as $\Delta \to 0$.

The delta function does not look like any function we have seen before, its values being either infinite or zero. It is therefore useful to view it as the limit of a more conventional function. Consider a Gaussian

$$g_\Delta(x - x') = \frac{1}{(\pi\Delta^2)^{1/2}} \exp\left[-\frac{(x - x')^2}{\Delta^2}\right] \tag{1.10.19}$$

as shown in Fig. 1.8a). The Gaussian is centered at $x' = x$, has width Δ, maximum height $(\pi\Delta^2)^{-1/2}$, and unit area, independent of Δ. As Δ approaches zero, g_Δ becomes a better and better approximation to the delta function.‡

It is obvious from the Gaussian model that the delta function is even. This may be verified as follows:

$$\delta(x - x') = \langle x \mid x' \rangle = \langle x' \mid x \rangle^* = \delta(x' - x)^* = \delta(x' - x)$$

since the delta function is real.

Consider next an object that is even more peculiar than the delta function: its derivative with respect to the *first* argument x:

$$\delta'(x - x') = \frac{d}{dx}\,\delta(x - x') = -\frac{d}{dx'}\,\delta(x - x') \tag{1.10.20}$$

What is the action of this function under the integral? The clue comes from the Gaussian model. Consider $dg_\Delta(x - x')/dx = -dg_\Delta(x - x')/dx'$ as a function of x'. As g_Δ shrinks, each bump at $\pm\varepsilon$ will become, up to a scale

‡ A fine point that will not concern you till Chapter 8: This formula for the delta function is valid even if Δ^2 is pure imaginary, say, equal to $i\beta^2$. First we see from Eq. (A.2.5) that g has unit area. Consider next the integral of g times $f(x')$ over a region in x' that includes x. For the most part, we get zero because f is smooth and g is wildly oscillating as $\beta \to 0$. However, at $x = x'$, the derivative of the phase of g vanishes and the oscillations are suspended. Pulling $f(x' = x)$ out of the integral, we get the desired result.

factor, the δ function. The first one will sample $-f(x-\varepsilon)$ and the second one $+f(x+\varepsilon)$, again up to a scale, so that

$$\int \delta'(x-x')f(x')\,dx' \propto f(x+\varepsilon) - f(x-\varepsilon) = 2\varepsilon \left.\frac{df}{dx'}\right|_{x'=x}$$

The constant of proportionality happens to be $1/2\varepsilon$ so that

$$\int \delta'(x-x')f(x')\,dx' = \left.\frac{df}{dx'}\right|_{x'=x} = \frac{df(x)}{dx} \tag{1.10.21}$$

This result may be verified as follows:

$$\int \delta'(x-x')f(x')\,dx' = \int \frac{d\delta(x-x')}{dx} f(x')\,dx' = \frac{d}{dx}\int \delta(x-x')f(x')\,dx'$$
$$= \frac{d}{dx} f(x)$$

Note that $\delta'(x-x')$ is an odd function. This should be clear from Fig. 1.8b or Eq. (1.10.20). An equivalent way to describe the action of the δ' function is by the equation

$$\delta'(x-x') = \delta(x-x')\frac{d}{dx'} \tag{1.10.22}$$

where it is understood that both sides appear in an integral over x' and that the differential operator acts on any function that accompanies the δ' function in the integrand. In this notation we can describe the action of higher derivatives of the delta function:

$$\frac{d^n\delta(x-x')}{dx^n} = \delta(x-x')\frac{d^n}{dx'^n} \tag{1.10.23}$$

We will now develop an alternate representation of the delta function. We know from basic Fourier analysis that, given a function $f(x)$, we may define its transform

$$f(k) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \tag{1.10.24}$$

and its inverse

$$f(x') = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{ikx'} f(k)\,dk \tag{1.10.25}$$

Feeding Eq. (1.10.24) into Eq. (1.10.25), we get

$$f(x') = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x'-x)} \right) f(x)\, dx$$

Comparing this result with Eq. (1.10.18), we see that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x'-x)} = \delta(x'-x) \tag{1.10.26}$$

*Exercise 1.10.1.** Show that $\delta(ax) = \delta(x)/|a|$. [Consider $\int \delta(ax)\, d(ax)$. Remember that $\delta(x) = \delta(-x)$.]

*Exercise 1.10.2.** Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|}$$

where x_i are the zeros of $f(x)$. Hint: Where does $\delta(f(x))$ blow up? Expand $f(x)$ near such points in a Taylor series, keeping the first nonzero term.

*Exercise 1.10.3.** Consider the *theta function* $\theta(x - x')$ which vanishes if $x - x'$ is negative and equals 1 if $x - x'$ is positive. Show that the theta function is the integral of the delta function.

Operators in Infinite Dimensions

Having acquainted ourselves with the elements of this function space, namely, the kets $|f\rangle$ and the basis vectors $|x\rangle$, let us turn to the (linear) operators that act on them. Consider the equation

$$\Omega |f\rangle = |\tilde{f}\rangle$$

Since the kets are in correspondence with the functions, Ω takes the function $f(x)$ into another, $\tilde{f}(x)$. Now, one operator that does such a thing is the familiar differential operator, which, acting on $f(x)$, gives $\tilde{f}(x) = df(x)/dx$. In the function space we can describe the action of this operator as

$$D|f\rangle = |df/dx\rangle$$

where $|df/dx\rangle$ is the ket corresponding to the function df/dx. What are the matrix elements of D in the $|x\rangle$ basis? To find out, we dot both sides of the above equation with $|x\rangle$,

$$\langle x | D | f\rangle = \left\langle x \middle| \frac{df}{dx} \right\rangle = \frac{df(x)}{dx}$$