Compatible Operators : $\rightarrow$

Consider the generalized uncertainty relationship Eq. (3.62).

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Two operators are said to be compatible if their commutator vanishes

i.e. $[\hat{A}, \hat{B}] = 0 \Leftrightarrow$ compatible

operators. Obviously $[\hat{X}, \hat{P}] = i\hbar \neq 0$

$\Rightarrow$ $\hat{X}$ and $\hat{P}$ are not compatible. For compatible operators then

$$\sigma_A \sigma_B \geq 0$$

This means we can construct states $\ni$ both operators can be precisely measured in these states.

In general it can be proved that for compatible operators we can find a common basis for both. The basis is labeled by two quantum numbers $\ni$

$$\hat{A}\, \phi_{a,b}(x) = a\, \phi_{a,b}(x) \;\&\; \hat{B}\, \phi_{a,b}(x) = b\, \phi_{a,b}(x).$$

N operators $\hat{\Omega}_1, \hat{\Omega}_2, \ldots \hat{\Omega}_N$ are said to be compatible iff

$$[\hat{\Omega}_i, \hat{\Omega}_j] = 0 \quad , \forall i, j = 1, 2, \ldots N$$

Then we can find a basis with N indices $\ni$

$$\hat{\Omega}_i \, \phi_{\omega_1, \omega_2, \ldots \omega_N}(x) = \omega_i \, \phi_{\omega_1, \omega_2, \ldots \omega_N}(x)$$

$$\forall i = 1, 2, \ldots N.$$

Find example in 1D.

Angular Momentum :→

Classically $\bar{L} = \bar{r} \times \bar{p}$

So in QM $\hat{\bar{L}} \equiv \hat{\bar{R}} \times \hat{\bar{P}}$

$$\Rightarrow \hat{L}_x = \cancel{x\,-i\hbar\partial}\; y\left(-i\hbar\frac{\partial}{\partial z}\right) - z(-i\hbar)\frac{\partial}{\partial y}$$

$$= i\hbar\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)$$

$$\hat{L}_y = i\hbar\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right), \quad \hat{L}_z = i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = i\hbar L_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar L_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar L_y.$$

Similarly $[\hat{L}^2, L_x] = 0$, $[\hat{L}^2, \hat{L}_y] = 0$, $[\hat{L}^2, \hat{L}_z] = 0$

∴ $\hat{L}^2$ and $\hat{L}_z$ are compatible

Hence we may find a basis ∋

$$\hat{L}^2 f = \lambda f \quad \text{and} \quad \hat{L}_z f = \mu f$$

where λ and μ are eigenvalues.

Define raising $\hat{L}_+$ and lowering $\hat{L}_-$ operators

$$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$$

$$\therefore \left[\hat{L}_z, \hat{L}_\pm\right] = \left[\hat{L}_z, \hat{L}_x\right] \pm i\left[\hat{L}_z, \hat{L}_y\right]$$

$$= i\hbar\hat{L}_y \pm i(-i\hbar\hat{L}_x) = \pm\hbar L_\pm \longrightarrow (4.106)$$

Also $\left[\hat{L}^2, \hat{L}_\pm\right] = 0$

$$\therefore \quad \hat{L}^2(\hat{L}_\pm f) = \lambda(\hat{L}_\pm f)$$

$\Rightarrow$ if f is an eigenfunction of $\hat{L}^2$ so also is $\hat{L}_\pm f$ and all 3 have the same eigenvalue λ.

Now $\hat{L}_z(L_\pm f) = \hat{L}_\pm \hat{L}_z f \pm \hbar \hat{L}_\pm f$ using Eq. (4.106)

$$\Rightarrow \hat{L}_z(\hat{L}_\pm f) = (\mu \pm \hbar)(\hat{L}_\pm f)$$

$$\Rightarrow \hat{L}_z(\hat{L}_\pm^n f) = (\mu \pm n\hbar)(\hat{L}_\pm f)$$

There must exist a limit on n on the lower as well as upper side

$$\Rightarrow \exists f_t \ni L_+ f_t = 0 \quad \text{and}$$

$$\exists f_b \ni L_- f_b = 0$$

Let $\hat{L}_z f_t = l\hbar f_t$, $\hat{L}^2 f_t = \lambda f_t$

Now $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}_\pm \hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z$

where we used $\hat{L}_\pm \hat{L}_\mp = \hat{L}_x^2 + \hat{L}_y^2 \mp i\hbar \hat{L}_z$

$$\therefore \quad \hat{L}^2 f_t = \left(\hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar \hat{L}_z\right) f_t = \hbar^2 l(l+1) f_t$$ 5

$$\Rightarrow \lambda = \hbar^2 l(l+1).$$

Now consider $\hat{L}_- f_b = 0$, $\hat{L}_z f_b = \hbar \bar{l} f_b$

$$\hat{L}^2 f_b = \lambda f_b$$

$$\Rightarrow \left(\hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z\right) f_b = \hbar^2 \bar{l}(\bar{l}-1) f_b$$

$$\Rightarrow \lambda = \hbar^2 l(l+1) = \hbar^2 \bar{l}(\bar{l}-1)$$

$\Rightarrow \bar{l} = l+1$ which is impossible since $\bar{l} < l$ or $\bar{l} = -l$

Now $\hat{L}_z f = \mu f \Rightarrow \mu = m\hbar \ni -l \leq m \leq l$

or in other words $m = -l, -l+1, \ldots l-1, l$
These are in general N steps $\ni$

$l = -l + N \Rightarrow l = N/2$ where N is a non-negative integer

$$\Rightarrow l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2} \ldots$$

$$m = -l, -l+1, \ldots l-1, l.$$

and

$$\hat{L}^2 f_l^m = \hbar^2 l(l+1) f_l^m, \quad \hat{L}_z f_l^m = m\hbar f_l^m.$$

Functional form of $\hat{L}^2, \hat{L}_\pm, \hat{L}_z, Y_l^m$.

$$\hat{\bar{L}} = \hat{\bar{R}} \times \hat{\bar{P}} = -i\hbar \hat{\bar{R}} \times \bar{\nabla}$$

$$\bar{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

in spherical polar coordinates.

Note $\hat{\bar{R}} \equiv \bar{r} = r\hat{r}$ and

$$\hat{r}\times\hat{r} = \hat{\theta}\times\hat{\theta} = \hat{\phi}\times\hat{\phi} = 0 \quad \text{and}$$

$$\hat{r}\times\hat{\theta} = \hat{\phi}, \quad \hat{\theta}\times\hat{\phi} = \hat{r}, \quad \hat{\phi}\times\hat{r} = \hat{\theta}.$$

$$\Rightarrow \hat{\bar{L}} = -i\hbar\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\therefore \hat{L}_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{L}_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}^2 = \frac{-\hbar^2}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Note then that
$$\hat{L}^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

and $\hat{L}_z f_\ell^m = m\hbar f_\ell^m$
$$\Rightarrow f_\ell^m \equiv Y_\ell^m(\theta, \phi).$$

$\Rightarrow Y_\ell^m$ are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$.

Now remember that for a central potential for which $V(\bar{r}) = V(r)$ we get

$$\hat{H} = \frac{-\hbar^2}{2m} \nabla^2 + V(r)$$
$$= \frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\right] + V(r) + \frac{\hat{L}^2}{2mr^2}$$
$$\Rightarrow \left[\hat{H}, \hat{L}^2\right] = 0$$

Also $\left[\hat{H}, \hat{L}_z\right] = 0$

Hence for central potentials we can always find $\Psi_{n\ell m}$ $\ni$

$$\hat{H}\Psi_{n\ell m} = E_n \Psi_{n\ell m}, \quad \hat{L}^2 \Psi_{n\ell m} = \hbar^2 \ell(\ell+1)\Psi_{n\ell m}$$

$$\hat{L}_z \Psi_{n\ell m} = \hbar m \Psi_{n\ell m}$$