Problem 4.15

(a) $\Psi(\mathbf{r},t) = \frac{1}{\sqrt{2}}\left(\psi_{2\,1\,1}e^{-iE_2t/\hbar} + \psi_{2\,1\,-1}e^{-iE_2t/\hbar}\right) = \frac{1}{\sqrt{2}}\left(\psi_{2\,1\,1} + \psi_{2\,1\,-1}\right)e^{-iE_2t/\hbar}; \quad E_2 = \frac{E_1}{4} = -\frac{\hbar^2}{8ma^2}$.

From Problem 4.11(b):

$$\psi_{2\,1\,1} + \psi_{2\,1\,-1} = -\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}re^{-r/2a}\sin\theta\left(e^{i\phi} - e^{-i\phi}\right) = -\frac{i}{\sqrt{\pi a}\,4a^2}re^{-r/2a}\sin\theta\,\sin\phi.$$

$$\boxed{\Psi(\mathbf{r},t) = -\frac{i}{\sqrt{2\pi a}\,4a^2}re^{-r/2a}\sin\theta\,\sin\phi\,e^{-iE_2t/\hbar}.}$$

(b)

$$\langle V\rangle = \int|\Psi|^2\left(-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right)d^3\mathbf{r} = \frac{1}{(2\pi a)(16a^4)}\left(-\frac{e^2}{4\pi\epsilon_0}\right)\int\left(r^2e^{-r/a}\sin^2\theta\,\sin^2\phi\right)\frac{1}{r}r^2\sin\theta\,dr\,d$$

$$= \frac{1}{32\pi a^5}\left(-\frac{\hbar^2}{ma^2}\right)\int_0^\infty r^3e^{-r/a}\,dr\int_0^\pi\sin^3\theta\,d\theta\int_0^{2\pi}\sin^2\phi\,d\phi = -\frac{\hbar^2}{32\pi ma^6}\left(3!a^4\right)\left(\frac{4}{3}\right)(\pi)$$

$$= \boxed{-\frac{\hbar^2}{4ma^2} = \frac{1}{2}E_1 = \frac{1}{2}(-13.6\text{eV}) = -6.8\text{eV}} \quad \text{(independent of } t\text{).}$$

4.15(b) Smart way to answer!

$$\langle V \rangle = \int \left(\frac{-e^2}{4\pi\epsilon_0 r}\right)\left(\frac{1}{2}\right) |\psi_{211} + \psi_{21-1}|^2 d^3\bar{r}$$

Note $\psi_{211} + \psi_{21-1} = R_{21}(r)\left[Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right]$

Now $d^3\bar{r} = r^2 dr\, d\Omega$

where $d\Omega = \sin\theta\, d\theta\, d\phi$

Also $\int (Y_{l'}^{m'})^* Y_l^m d\Omega = \delta_{l,l'}\delta_{m,m'}$

$$\Rightarrow \int d\Omega |\psi_{211} + \psi_{21-1}|^2 = |R_{21}|^2 [1+1+0+0]$$

$$= 2|R_{21}|^2$$

$$\Rightarrow \langle V \rangle = \frac{-e^2}{4\pi\epsilon_0} \int_0^\infty |R_{21}|^2 r\, dr$$

$$= \frac{-e^2}{4\pi\epsilon_0} \times \frac{1}{24a^5} \int_0^\infty r^3 e^{-r/a} dr \quad \text{(using Table 4.7)}$$

$$= \frac{-e^2}{4\pi\epsilon_0} \times \frac{1}{24a} \int_0^\infty x^3 e^{-x} dx$$

$$= \left[\frac{-e^2}{4\pi\epsilon_0 a}\right] \times \frac{1}{4} = \frac{E_1}{2} = \frac{-13.6\,eV}{2} = -6.8\,eV$$

Problem 4.18

$$\langle f|L_\pm g\rangle = \langle f|L_x g\rangle \pm i\langle f|L_y g\rangle = \langle L_x f|g\rangle \pm i\langle L_y f|g\rangle = \langle (L_x \mp iL_y)f|g\rangle = \langle L_\mp f|g\rangle, \text{ so } (L_\pm)^\dagger = L_\mp.$$

Now, using Eq. 4.112, in the form $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$:

$$\begin{aligned}
\langle f_l^m|L_\mp L_\pm f_l^m\rangle &= \langle f_l^m|(L^2 - L_z^2 \mp \hbar L_z)f_l^m\rangle = \langle f_l^m|\left[\hbar^2 l(l+1) - \hbar^2 m^2 \mp \hbar^2 m\right] f_l^m\rangle \\
&= \hbar^2\left[l(l+1) - m(m\pm 1)\right]\langle f_l^m|f_l^m\rangle = \hbar^2\left[l(l+1) - m(m\pm 1)\right] \\
&= \langle L_\pm f_l^m|L_\pm f_l^m\rangle = \langle A_l^m f_l^{m\pm 1}|A_l^m f_l^{m\pm 1}\rangle = |A_l^m|^2\langle f_l^{m\pm 1}|f_l^{m\pm 1}\rangle = |A_l^m|^2.
\end{aligned}$$

Problem 4.21

(a)

$$
\begin{aligned}
L_+L_-f &= -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right] \\
&= -\hbar^2 e^{i\phi}\left\{e^{-i\phi}\left[\frac{\partial^2 f}{\partial\theta^2} - i\left(-\csc^2\theta\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi}\right)\right]\right. \\
&\quad \left. + i\cot\theta\left[-ie^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right) + e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\,\partial\theta} - i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right]\right\} \\
&= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} - i\cot^2\theta\frac{\partial f}{\partial\phi} + i\cot\theta\frac{\partial^2 f}{\partial\phi\,\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i(\csc^2\theta - \cot^2\theta)\frac{\partial}{\partial\phi}\right]f, \quad \text{so} \\
L_+L_- &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right). \quad \text{QED}
\end{aligned}
$$

(b) Equation 4.129 $\Rightarrow L_z = \dfrac{\hbar}{i}\dfrac{\partial}{\partial\phi}$, Eq. 4.112 $\Rightarrow L^2 = L_+L_- + L_z^2 - \hbar L_z$, so, using (a):

$$
\begin{aligned}
L^2 &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} - \hbar\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial\phi} \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + (\cot^2\theta + 1)\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial\phi}\right) = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \quad \text{QED}
\end{aligned}
$$