Problem 4.2

(a) Equation 4.8 $\Rightarrow -\dfrac{\hbar^2}{2m}\left(\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}+\dfrac{\partial^2\psi}{\partial z^2}\right) = E\psi$ (inside the box). Separable solutions:
$X(x)Y(y)Z(z)$. Put this in, and divide by XYZ:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2X}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2} = -\frac{2m}{\hbar^2}E.$$

The three terms on the left are functions of x, y, and z, respectively, so each must be a consta
separation constants k_x^2, k_y^2, and k_z^2 (as we'll soon seen, they must be positive).

$$\frac{d^2X}{dx^2} = -k_x^2X; \quad \frac{d^2Y}{dy^2} = -k_y^2Y; \quad \frac{d^2Z}{dz^2} = -k_z^2Z, \quad \text{with} \quad E = \frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2).$$

Solution:

$$X(x) = A_x\sin k_xx + B_x\cos k_xx; \quad Y(y) = A_y\sin k_yy + B_y\cos k_yy; \quad Z(z) = A_z\sin k_zz +$$

But $X(0)=0$, so $B_x=0$; $Y(0)=0$, so $B_y=0$; $Z(0)=0$, so $B_z=0$. And $X(a)=0 \Rightarrow$ sin
$k_x = n_x\pi/a$ ($n_x = 1,2,3,\dots$). [As before (page 31), $n_x \neq 0$, and negative values are redunda
$k_y = n_y\pi/a$ and $k_z = n_z\pi/a$. So

$$\psi(x,y,z) = A_xA_yA_z\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right), \quad E = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}(n_x^2+n_y^2+$$

We might as well normalize X, Y, and Z separately: $A_x = A_y = A_z = \sqrt{2/a}$. *Conclusion:*

$$\boxed{\psi(x,y,z) = \left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right); \quad E = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2+n_y^2+n_z^2); \; n_}$$

(b)

n_x	n_y	n_z	$(n_x^2+n_y^2+n_z^2)$
1	1	1	3
1	1	2	6
1	2	1	6
2	1	1	6
1	2	2	9
2	1	2	9
2	2	1	9
1	1	3	11
1	3	1	11
3	1	1	11
2	2	2	12
1	2	3	14
1	3	2	14
2	1	3	14
2	3	1	14
3	1	2	14
3	2	1	14

Energy	Degeneracy
$E_1 = 3\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 1$
$E_2 = 6\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 3$.
$E_3 = 9\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 3$.
$E_4 = 11\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 3$.
$E_5 = 12\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 1$.
$E_6 = 14\frac{\pi^2\hbar^2}{2ma^2}$;	$d = 6$.

(c) The next combinations are: $E_7(322)$, $E_8(411)$, $E_9(331)$, $E_{10}(421)$, $E_{11}(332)$, $E_{12}(422)$, $E_{13}(431)$, and $E_{14}(333 \text{ and } 511)$. The degeneracy of E_{14} is $\boxed{4.}$ Simple combinatorics accounts for degeneracies of 1 ($n_x = n_y = n_z$), 3 (two the same, one different), or 6 (all three different). But in the case of E_{14} there is a numerical "accident": $3^2 + 3^2 + 3^2 = 27$, but $5^2 + 1^2 + 1^2$ is *also* 27, so the degeneracy is greater than combinatorial reasoning alone would suggest.

Problem 4.4

$$\frac{d\Theta}{d\theta} = \frac{A}{\tan(\theta/2)}\frac{1}{2}\sec^2(\theta/2) = \frac{A}{2}\frac{1}{\sin(\theta/2)\cos(\theta/2)} = \frac{A}{\sin\theta}. \quad \text{Therefore} \quad \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}(A)$$

With $l = m = 0$, Eq. 4.25 reads: $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$. So $A\ln[\tan(\theta/2)]$ *does* satisfy Eq. 4.25.

$$\Theta(0) = A\ln(0) = A(-\infty);\ \Theta(\pi) = A\ln\left(\tan\frac{\pi}{2}\right) = A\ln(\infty) = A(\infty). \quad \boxed{\Theta \text{ blows up at } \theta = 0 \text{ and}}$$

Problem 4.5

$$Y_l^l = (-1)^l\sqrt{\frac{(2l+1)}{4\pi}\frac{1}{(2l)!}}\,e^{il\phi}P_l^l(\cos\theta). \quad P_l^l(x) = (1-x^2)^{l/2}\left(\frac{d}{dx}\right)^l P_l(x).$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l, \quad \text{so } P_l^l(x) = \frac{1}{2^l l!}(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^{2l}(x^2-1)^l.$$

Now $(x^2-1)^l = x^{2l} + \cdots$, where all the other terms involve powers of x *less* than $2l$, and hence give differentiated $2l$ times. So

$$P_l^l(x) = \frac{1}{2^l l!}(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^{2l}x^{2l}. \quad \text{But } \left(\frac{d}{dx}\right)^n x^n = n!, \quad \text{so } P_l^l = \frac{(2l)!}{2^l l!}(1-x^2)^{l/2}.$$

$$\therefore Y_l^l = (-1)^l\sqrt{\frac{(2l+1)}{4\pi(2l)!}}\,e^{il\phi}\frac{(2l)!}{2^l l!}(\sin\theta)^l = \boxed{\frac{1}{l!}\sqrt{\frac{(2l+1)!}{4\pi}}\left(-\frac{1}{2}e^{i\phi}\sin\theta\right)^l.}$$