Problem 3.33

Equation 2.69: $x = \sqrt{\dfrac{\hbar}{2m\omega}}(a_+ + a_-), \quad p = i\sqrt{\dfrac{\hbar m\omega}{2}}(a_+ - a_-);$ Eq. 2.66 : $\begin{cases} a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle, \\ a_-|n\rangle = \sqrt{n}\,|n-1\rangle. \end{cases}$

$$\langle n|x|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a_+ + a_-)|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n'+1}\,\langle n|n'+1\rangle + \sqrt{n'}\,\langle n|n'-1\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'+1}\,\delta_{n,n'+1} + \sqrt{n'}\,\delta_{n,n'-1}\right) = \boxed{\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n'}\,\delta_{n,n'-1}\right).}$$

$$\langle n|p|n'\rangle = \boxed{i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n}\,\delta_{n',n-1} - \sqrt{n'}\,\delta_{n,n'-1}\right).}$$

Noting that n and n' run from zero to infinity, the matrices are:

$$\boxed{\mathsf{X} = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & \vdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & \\ & & & \cdots & & & \end{pmatrix}; \quad \mathsf{P} = i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & \vdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & 0 & \\ 0 & 0 & 0 & \sqrt{4} & & -\sqrt{5} & \\ & & & \cdots & & & \end{pmatrix}.}$$

Squaring these matrices:

$$\mathsf{X}^2 = \frac{\hbar}{2m\omega}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & 3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & 5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix};$$

$$\mathsf{P}^2 = -\frac{m\hbar\omega}{2}\begin{pmatrix} -1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & -3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & -5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & -7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix}.$$

Problem 3.35

(a) $\langle x\rangle = \langle\alpha|x\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|(a_+ + a_-)\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle a_-\alpha|\alpha\rangle + \langle\alpha|a_-\alpha\rangle\right) = \boxed{\sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*).}$

$$x^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right). \quad \text{But } a_-a_+ = [a_-, a_+] + a_+a_- = 1 + a_+a_- \quad \text{(Eq. 2.55).}$$

$$= \frac{\hbar}{2m\omega}\left(a_+^2 + 2a_+a_- + 1 + a_-^2\right).$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\alpha|\left(a_+^2 + 2a_+a_- + 1 + a_-^2\right)\alpha\rangle = \frac{\hbar}{2m\omega}\left(\langle a_-^2\alpha|\alpha\rangle + 2\langle a_-\alpha|a_-\alpha\rangle + \langle\alpha|\alpha\rangle + \langle\alpha|a_-^2\alpha\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2 + 2(\alpha^*)\alpha + 1 + \alpha^2\right] = \boxed{\frac{\hbar}{2m\omega}\left[1 + (\alpha+\alpha^*)^2\right].}$$

$$\langle p\rangle = \langle\alpha|p\alpha\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle\alpha|(a_+ - a_-)\alpha\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle a_-\alpha|\alpha\rangle - \langle\alpha|a_-\alpha\rangle\right) = \boxed{-i\sqrt{\frac{\hbar m\omega}{2}}(\alpha-\alpha^*).}$$

$$p^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right) = -\frac{\hbar m\omega}{2}\left(a_+^2 - 2a_+a_- - 1 + a_-^2\right).$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\langle\alpha|\left(a_+^2 - 2a_+a_- - 1 + a_-^2\right)\alpha\rangle = -\frac{\hbar m\omega}{2}\left(\langle a_-^2\alpha|\alpha\rangle - 2\langle a_-\alpha|a_-\alpha\rangle - \langle\alpha|\alpha\rangle + \langle\alpha|a_-^2\alpha\rangle\right)$$

$$= -\frac{\hbar m\omega}{2}\left[(\alpha^*)^2 - 2(\alpha^*)\alpha - 1 + \alpha^2\right] = \boxed{\frac{\hbar m\omega}{2}\left[1 - (\alpha-\alpha^*)^2\right].}$$

(b)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}\left[1 + (\alpha+\alpha^*)^2 - (\alpha+\alpha^*)^2\right] = \frac{\hbar}{2m\omega};$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar m\omega}{2}\left[1 - (\alpha-\alpha^*)^2 + (\alpha-\alpha^*)^2\right] = \frac{\hbar m\omega}{2}. \quad \sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}. \quad \text{QED}$$

(c) Using Eq. 2.67 for ψ_n:

$$c_n = \langle\psi_n|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle(a_+)^n\psi_0|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle\psi_0|(a_-)^n\alpha\rangle = \frac{1}{\sqrt{n!}}\alpha^n\langle\psi_0|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}c_0. \quad ✓$$

(d) $1 = \sum_{n=0}^{\infty}|c_n|^2 = |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \quad \Rightarrow \quad \boxed{c_0 = e^{-|\alpha|^2/2}.}$

(e) $|\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}e^{-i(n+\frac{1}{2})\omega t}|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}}e^{-|\alpha|^2/2}|n\rangle.$

Apart form the overall phase factor $e^{-i\omega t/2}$ (which doesn't affect its status as an eigenfunction of a_-, or its eigenvalue), $|\alpha(t)\rangle$ is the same as $|\alpha\rangle$, but with eigenvalue $\alpha(t) = e^{-i\omega t}\alpha$. ✓

(f) Equation 2.58 says $a_-|\psi_0\rangle = 0$, so $\boxed{\text{yes,}}$ it *is* a coherent state, with eigenvalue $\boxed{\alpha = 0.}$