Problem 2.34

(a)

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar}; \ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

(b)

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar}; \ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = ilF$.

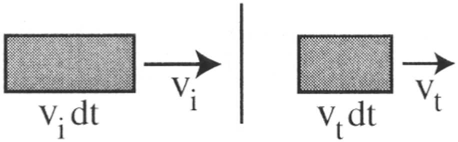
$$\Rightarrow A + B = \frac{k}{l}(A - B); \ A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1 - k/l)^2}{(1 + k/l)^2} = \frac{(k - l)^2}{(k + l)^2} = \frac{(k - l)^4}{(k^2 - l^2)^2}.$$

Now $k^2 - l^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \left(\frac{2m}{\hbar^2}\right)V_0; \ k - l = \frac{\sqrt{2m}}{\hbar}[\sqrt{E} - \sqrt{E - V_0}]$, so

$$\boxed{R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}.}$$

(c)



From the diagram, $T = P_t/P_i = |F|^2 v_t/|A|^2 v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\frac{v_t}{v_i} = \frac{\sqrt{E-V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T = \sqrt{\frac{E-V_0}{E}}\left|\frac{F}{A}\right|^2$. Alternatively, from Problem 2.19:

$$J_i = \frac{\hbar k}{m}|A|^2; \quad J_t = \frac{\hbar l}{m}|F|^2; \quad T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E-V_0}{E}}.$$

For $E < V_0$, of course, $\boxed{T = 0.}$

(d)

$$\text{For } E > V_0, \ F = A + B = A + A\frac{\left(\frac{k}{l}-1\right)}{\left(\frac{k}{l}+1\right)} = A\frac{2k/l}{\left(\frac{k}{l}+1\right)} = \frac{2k}{k+l}A.$$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2-l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E-V_0}(\sqrt{E}-\sqrt{E-V_0})^2}{V_0^2}}.$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2+2kl+l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1. \checkmark$$

Problem 2.45

$$\left.\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1 \Rightarrow -\frac{\hbar^2}{2m}\psi_2\frac{d^2\psi_1}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2 \Rightarrow -\frac{\hbar^2}{2m}\psi_1\frac{d^2\psi_2}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \end{aligned}\right\} \Rightarrow -\frac{\hbar^2}{2m}\left[\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}\right]$$

But $\frac{d}{dx}\left[\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right] = \frac{d\psi_2}{dx}\frac{d\psi_1}{dx} + \psi_2\frac{d^2\psi_1}{dx^2} - \frac{d\psi_1}{dx}\frac{d\psi_2}{dx} - \psi_1\frac{d^2\psi_2}{dx^2} = \psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}$. S

zero, it follows that $\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = K$ (a constant). But $\psi \to 0$ at ∞ so the constant must be zero

$\psi_2\frac{d\psi_1}{dx} = \psi_1\frac{d\psi_2}{dx}$, or $\frac{1}{\psi_1}\frac{d\psi_1}{dx} = \frac{1}{\psi_2}\frac{d\psi_2}{dx}$, so $\ln\psi_1 = \ln\psi_2 +$ constant, or $\psi_1 = (\text{constant})\psi_2$. QED