Problem 2.22

(a)

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2}\,dx = |A|^2\sqrt{\frac{\pi}{2a}};\qquad \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}.}$$

(b)

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)}\frac{1}{\sqrt{a}}\,dy = \frac{1}{\sqrt{a}}e^{b^2/4a}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\frac{\pi}{a}}\,e^{b^2/4a}.$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}A\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{\pi}{a}}\,e^{-k^2/4a} = \frac{1}{(2\pi a)^{1/4}}e^{-k^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty}\underbrace{e^{-k^2/4a}e^{i(kx-\hbar k^2t/2m)}}_{e^{-[(\frac{1}{4a}+i\hbar t/2m)k^2-ixk]}}\,dk$$

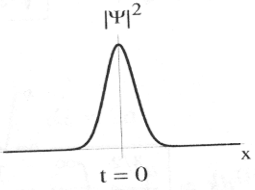
$$= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a}+i\hbar t/2m}}e^{-x^2/4(\frac{1}{4a}+i\hbar t/2m)} = \boxed{\left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}.}$$

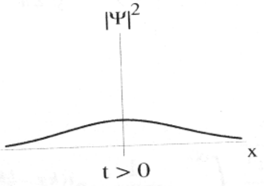
(c)

Let $\theta \equiv 2\hbar at/m$. Then $|\Psi|^2 = \sqrt{\frac{2a}{\pi}}\frac{e^{-ax^2/(1+i\theta)}e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$$-\frac{ax^2}{(1+i\theta)} - \frac{ax^2}{(1-i\theta)} = -ax^2\frac{(1-i\theta+1+i\theta)}{(1+i\theta)(1-i\theta)} = \frac{-2ax^2}{1+\theta^2};\quad |\Psi|^2 = \sqrt{\frac{2a}{\pi}}\frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}.$$

Or, with $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}}we^{-2w^2x^2}.}$ As t increases, the graph of $|\Psi|^2$ flattens out and broadens.





Problem 2.24

(a) Let $y \equiv cx$, so $dx = \frac{1}{c}dy$. $\left\{\begin{array}{l}\text{If } c > 0,\ y : -\infty \to \infty. \\ \text{If } c < 0,\ y : \infty \to -\infty.\end{array}\right\}$

$$\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \begin{cases} \frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = \frac{1}{c}f(0) \quad (c > 0);\ \text{or} \\ \frac{1}{c}\int_{\infty}^{-\infty} f(y/c)\delta(y)dy = -\frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = -\frac{1}{c}f(0)\ (c < 0). \end{cases}$$

In either case, $\displaystyle\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \frac{1}{|c|}f(0) = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)dx$. So $\delta(cx) = \frac{1}{|c|}\delta(x)$. ✓

(b)

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx}\theta dx \quad \text{(integration by parts)}$$

$$= f(\infty) - \int_{0}^{\infty} \frac{df}{dx}dx = f(\infty) - f(\infty) + f(0) = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx.$$

So $d\theta/dx = \delta(x)$. ✓ [Makes sense: The θ function is constant (so derivative is zero) except at $x = 0$,
the derivative is infinite.]