Problem 3.8

(a) The eigenvalues (Eq. 3.29) are $0, \pm 1, \pm 2, \ldots$, which are obviously real. ✓ For any two eigenfunctions, $f = A_q e^{-iq\phi}$ and $g = A_{q'} e^{-iq'\phi}$ (Eq. 3.28), we have

$$\langle f|g\rangle = A_q^* A_{q'} \int_0^{2\pi} e^{iq\phi} e^{-iq'\phi}\, d\phi = A_q^* A_{q'} \left.\frac{e^{i(q-q')\phi}}{i(q-q')}\right|_0^{2\pi} = \frac{A_q^* A_{q'}}{i(q-q')}\left[e^{i(q-q')2\pi} - 1\right].$$

But q and q' are *integers*, so $e^{i(q-q')2\pi} = 1$, and hence $\langle f|g\rangle = 0$ (provided $q \neq q'$, so the denominator is nonzero). ✓

(b) In Problem 3.6 the eigenvalues are $q = -n^2$, with $n = 0, 1, 2, \ldots$, which are obviously real. ✓ For any two eigenfunctions, $f = A_q e^{\pm in\phi}$ and $g = A_{q'} e^{\pm in'\phi}$, we have

$$\langle f|g\rangle = A_q^* A_{q'} \int_0^{2\pi} e^{\mp in\phi} e^{\pm in'\phi}\, d\phi = A_q^* A_{q'} \left.\frac{e^{\pm i(n'-n)\phi}}{\pm i(n'-n)}\right|_0^{2\pi} = \frac{A_q^* A_{q'}}{\pm i(n'-n)}\left[e^{\pm i(n'-n)2\pi} - 1\right] = 0$$

(provided $n \neq n'$). But notice that for each eigenvalue (i.e. each value of n) there are *two* eigenfunctions (one with the plus sign and one with the minus sign), and these are *not* orthogonal to one another.

Problem 3.27

(a) $\boxed{\psi_1.}$

(b) $\boxed{b_1 \text{ (with probability } 9/25) \text{ or } b_2 \text{ (with probability } 16/25).}$

(c) Right after the measurement of B:

- With probability 9/25 the particle is in state $\phi_1 = (3\psi_1 + 4\psi_2)/5$; in that case the probability of getting a_1 is 9/25.
- With probability 16/25 the particle is in state $\phi_2 = (4\psi_1 - 3\psi_2)/5$; in that case the probability of getting a_1 is 16/25.

So the total probability of getting a_1 is $\frac{9}{25} \cdot \frac{9}{25} + \frac{16}{25} \cdot \frac{16}{25} = \boxed{\frac{337}{625} = 0.5392.}$

[*Note:* The measurment of B (even if we don't know the *outcome* of that measurement) collapses the wave function, and thereby alters the probabilities for the second measurment of A. If the graduate student inadvertantly neglected to measure B, the second measurement of A would be *certain* to reproduce the result a_1.]

Problem 3.2

(a)

$$\langle f|f\rangle = \int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1}x^{2\nu+1}\Big|_0^1 = \frac{1}{2\nu+1}\left(1-0^{2\nu+1}\right).$$

Now $0^{2\nu+1}$ is finite (in fact, *zero*) provided $(2\nu+1)>0$, which is to say, $\boxed{\nu > -\frac{1}{2}.}$ If $(2\nu+1)<0$ the integral definitely blows up. As for the critical case $\nu=-\frac{1}{2}$, this must be handled separately:

$$\langle f|f\rangle = \int_0^1 x^{-1}\,dx = \ln x\Big|_0^1 = \ln 1 - \ln 0 = 0+\infty.$$

So $f(x)$ is in Hilbert space only for ν strictly *greater* than -1/2.

(b) For $\nu = 1/2$, we know from (a) that $f(x)$ *is* in Hilbert space: $\boxed{\text{yes.}}$

Since $xf = x^{3/2}$, we know from (a) that it *is* in Hilbert space: $\boxed{\text{yes.}}$

For $df/dx = \frac{1}{2}x^{-1/2}$, we know from (a) that it is *not* in Hilbert space: $\boxed{\text{no.}}$

[*Moral:* Simple operations, such as differenting (or multiplying by $1/x$), can carry a function *out* of Hilbert space.]