Problem 5.20

Positive-energy solutions. These are the same as before, except that α (and hence also β) is now a negative number.

Negative-energy solutions. On $0 < x < a$ we have

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi, \quad \text{where} \quad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \quad \Rightarrow \quad \psi(x) = A\sinh kx + B\cosh kx.$$

According to Bloch's theorem the solution on $-a < x < 0$ is

$$\psi(x) = e^{-iKa}\left[A\sinh\kappa(x+a) + B\cosh\kappa(x+a)\right].$$

Continuity at $x = 0 \Rightarrow$

$$B = e^{-iKa}\left[A\sinh\kappa a + B\cosh\kappa a\right], \quad \text{or} \quad A\sinh\kappa a = B\left[e^{iKa} - \cosh\kappa a\right]. \tag{1}$$

The discontinuity in ψ' (Eq. 2.125) $\Rightarrow$

$$\kappa A - e^{-iKa}\kappa\left[A\cosh\kappa a + B\sinh\kappa a\right] = \frac{2m\alpha}{\hbar^2}B, \text{ or } A\left[1 - e^{-iKa}\cosh\kappa a\right] = B\left[\frac{2m\alpha}{\hbar^2\kappa} + e^{-iKa}\sinh\kappa a\right]. \tag{2}$$

Plugging (1) into (2) and cancelling B:

$$\left(e^{iKa} - \cosh\kappa a\right)\left(1 - e^{-iKa}\cosh\kappa a\right) = \frac{2m\alpha}{\hbar^2\kappa}\sinh\kappa a + e^{-iKa}\sinh^2\kappa a.$$

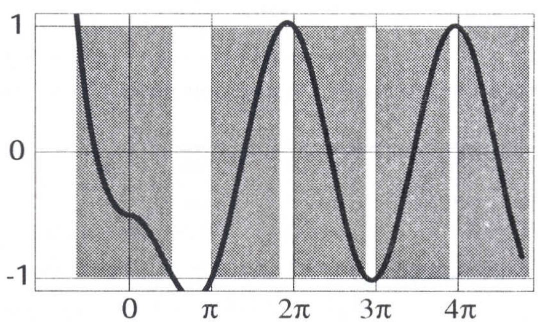
$$e^{iKa} - 2\cosh\kappa a + e^{-iKa}\cosh^2\kappa a - e^{-iKa}\sinh^2\kappa a = \frac{2m\alpha}{\hbar^2\kappa}\sinh\kappa a.$$

$$e^{iKa} + e^{-iKa} = 2\cosh\kappa a + \frac{2ma}{\hbar^2\kappa}\sinh\kappa a, \quad \boxed{\cos Ka = \cosh\kappa a + \frac{m\alpha}{\hbar^2\kappa}\sinh\kappa a.}$$

This is the analog to Eq. 5.64. As before, we let $\beta \equiv m\alpha a/\hbar^2$ (but remember it's now a *negative* number), and this time we define $z \equiv -\kappa a$, extending Eq. 5.65 to negative z, where it represents negative-energy solutions. In this region we define

$$f(z) = \cosh z + \beta\frac{\sinh z}{z}. \tag{3}$$

In the Figure I have plotted $f(z)$ for $\beta = -1.5$, using Eq. 5.66 for postive z and (3) for negative z. As before, allowed energies are restricted to the range $-1 \le f(z) \le 1$, and occur at intersections of $f(z)$ with the N horizontal lines $\cos Ka = \cos(2\pi n/Na)$, with $n = 0, 1, 2 \ldots N-1$. Evidently the first band (partly negative, and partly positive) contains N states, as do all the higher bands.



Problem 5.23

(a) $E_{n_1 n_2 n_3} = (n_1 + n_2 + n_3 + \frac{3}{2})\hbar\omega = \frac{9}{2}\hbar\omega \Rightarrow n_1 + n_2 + n_3 = 3. \quad (n_1, n_2, n_3 = 0, 1, 2, 3 \ldots).$

State n_1	State n_2	State n_3	Configuration $(N_0, N_1, N_2 \ldots)$	# of States
0	0	3		
0	3	0	(2,0,0,1,0,0 ...)	3
3	0	0		
0	1	2		
0	2	1		
1	0	2	(1,1,1,0,0,0 ...)	6
1	2	0		
2	0	1		
2	1	0		
1	1	1	(0,3,0,0,0 ...)	1

Possible single-particle energies:

$E_0 = \hbar\omega/2 : P_0 = 12/30 = 4/10.$
$E_1 = 3\hbar\omega/2 : P_1 = 9/30 = 3/10.$
$E_2 = 5\hbar\omega/2 : P_2 = 6/30 = 2/10.$
$E_3 = 7\hbar\omega/2 : P_3 = 3/30 = 1/10.$

Most probable configuration: (1,1,1,0,0,0 ...).

Most probable single-particle energy: $E_0 = \frac{1}{2}\hbar\omega.$

(b) For identical fermions the *only* configuration is (1,1,1,0,0,0 ...) (one state), so this is also the mos probable configuration. The possible one-particle energies are

$$E_0 \ (P_0 = 1/3), \quad E_1 \ (P_1 = 1/3), \quad E_2 \ (P_2 = 1/3),$$

and they are all equally likely, so it's a 3-way tie for the most probable energy.

(c) For identical bosons all three configurations are possible, and there is one state for each. Possible o particle energies: $E_0(P_0 = 1/3), E_1(P_1 = 4/9), E_2(P_2 = 1/9), E_3(P_3 = 1/9).$ Most probable energy: