Problem 3.37

First find the eigenvalues and eigenvectors of the Hamiltonian. The characteristic equation says

$$\begin{vmatrix} (a-E) & 0 & b \\ 0 & (c-E) & 0 \\ b & 0 & (a-E) \end{vmatrix} = (a-E)(c-E)(a-E) - b^2(c-E) = (c-E)\left[(a-E)^2 - b^2\right] = 0,$$

Either $E = c$, or else $(a-E)^2 = b^2 \Rightarrow E = a \pm b$. So the eigenvalues are

$$E_1 = c, \quad E_2 = a + b, \quad E_3 = a - b.$$

To find the corresponding eigenvectors, write

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

(1)

$$\left.\begin{aligned} a\alpha + b\gamma &= c\alpha \Rightarrow (a-c)\alpha + b\gamma = 0; \\ c\beta &= c\beta \qquad \text{(redundant)} \qquad ; \\ b\alpha + a\gamma &= c\gamma \Rightarrow (a-c)\gamma + b\alpha = 0. \end{aligned}\right\} \Rightarrow \left[(a-c)^2 - b^2\right]\alpha = 0.$$

So (excluding the degenerate case $a - c = \pm b$) $\alpha = 0$, and hence also $\gamma = 0$.

(2)

$$\begin{aligned} a\alpha + b\gamma &= (a+b)\alpha \Rightarrow & \alpha - \gamma &= 0; \\ c\beta &= (a+b)\beta \Rightarrow & \beta &= 0; \\ b\alpha + a\gamma &= (a+b)\gamma & &\text{(redundant)}. \end{aligned}$$

So $\alpha = \gamma$ and $\beta = 0$.

(3)

$$\begin{aligned} a\alpha + b\gamma &= (a-b)\alpha \Rightarrow & \alpha + \gamma &= 0; \\ c\beta &= (a-b)\beta \Rightarrow & \beta &= 0; \\ b\alpha + a\gamma &= (a-b)\gamma & &\text{(redundant).} \end{aligned}$$

So $\alpha = -\gamma$ and $\beta = 0$.

Conclusion: The (normalized) eigenvectors of H are

$$|s_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |s_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |s_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(a) Here $|\mathcal{S}(0)\rangle = |s_1\rangle$, so

$$|\mathcal{S}(t)\rangle = e^{-iE_1t/\hbar}|s_1\rangle = \boxed{e^{-ict/\hbar}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.}$$

(b)

$$\begin{aligned} |\mathcal{S}(0)\rangle &= \frac{1}{\sqrt{2}}\left(|s_2\rangle + |s_3\rangle\right). \\ |\mathcal{S}(t)\rangle &= \frac{1}{\sqrt{2}}\left(e^{-iE_2t/\hbar}|s_2\rangle + e^{-iE_3t/\hbar}|s_3\rangle\right) = \frac{1}{\sqrt{2}}\left[e^{-i(a+b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^{-i(a-b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right] \\ &= \frac{1}{2}e^{-iat/\hbar}\begin{pmatrix} e^{-ibt/\hbar} + e^{ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} - e^{ibt/\hbar} \end{pmatrix} = \boxed{e^{-iat/\hbar}\begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i\sin(bt/\hbar) \end{pmatrix}.} \end{aligned}$$

Problem 4.48

(a) Using Eqs. 3.64 and 4.122: $[A, B] = [x^2, L_z] = x[x, L_z] + [x, L_z]x = x(-i\hbar y) + (-i\hbar y)x = -2i\hbar xy$.

Equation 3.62 $\Rightarrow \sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}(-2i\hbar)\langle xy\rangle\right]^2 = \hbar^2\langle xy\rangle^2 \Rightarrow \boxed{\sigma_A \sigma_B \geq \hbar|\langle xy\rangle|.}$

(b) Equation 4.113 $\Rightarrow \langle B\rangle = \langle L_z\rangle = m\hbar;\quad \langle B^2\rangle = \langle L_z^2\rangle = m^2\hbar^2;$ so $\sigma_B = m^2\hbar^2 - m^2\hbar^2 = \boxed{0.}$

(c) Since the left side of the uncertainty principle is zero, the right side must also be: $\boxed{\langle xy\rangle = 0,}$ for eigenstates of L_z.