Problem 4.27

(a)

$$\chi^\dagger\chi = |A|^2(9+16) = 25|A|^2 = 1 \Rightarrow \boxed{A = 1/5.}$$

(b)

$$\langle S_x\rangle = \chi^\dagger \mathsf{S}_x\chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}4\\3i\end{pmatrix} = \frac{\hbar}{50}(12i+12i) = \boxed{0.}$$

$$\langle S_y\rangle = \chi^\dagger \mathsf{S}_y\chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}-4i\\-3\end{pmatrix} = \frac{\hbar}{50}(-12-12) = \boxed{-\frac{12}{25}\hbar.}$$

$$\langle S_z\rangle = \chi^\dagger \mathsf{S}_z\chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}3i\\-4\end{pmatrix} = \frac{\hbar}{50}(9-16) = \boxed{\frac{7}{50}\hbar.}$$

(c)

$$\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4} \text{ (always, for spin 1/2), so } \sigma_{S_x}^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0,\quad \boxed{\sigma_{S_x} = \frac{\hbar}{2}.}$$

$$\sigma_{S_y}^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar}{4} - \left(\frac{12}{25}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-576) = \frac{49}{2500}\hbar^2,\quad \boxed{\sigma_{S_y} = \frac{7}{50}\hbar.}$$

$$\sigma_{S_z}^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-49) = \frac{576}{2500}\hbar^2,\quad \boxed{\sigma_{S_z} = \frac{12}{25}\hbar.}$$

(d)

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_z\rangle| = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \quad \text{(right } \textit{at} \text{ the uncertainty limit). } \checkmark$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar\cdot\frac{12}{25}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_x\rangle| = 0 \quad \text{(trivial). } \checkmark$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar\cdot\frac{\hbar}{2} \overset{?}{\geq} \frac{\hbar}{2}|\langle S_y\rangle| = \frac{\hbar}{2}\cdot\frac{12}{25}\hbar \quad \text{(right } \textit{at} \text{ the uncertainty limit). } \checkmark$$

Problem 4.32

(a) Using Eqs. 4.151 and 4.163:

$$c_+^{(x)} = \chi_+^{(x)\dagger}\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right].$$

$$P_+^{(x)}(t) = |c_+^{(x)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$$
$$= \frac{1}{2}\left[\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}\right)\right]$$
$$= \frac{1}{2}\left[1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos(\gamma B_0 t)\right] = \boxed{\frac{1}{2}\left[1 + \sin\alpha\cos(\gamma B_0 t)\right].}$$

(b) From Problem 4.29(a): $\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$.

$$c_+^{(y)} = \chi_+^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right];$$

$$P_+^{(y)}(t) = |c_+^{(y)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + i\sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$$
$$= \frac{1}{2}\left[\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} - e^{-i\gamma B_0 t}\right)\right]$$
$$= \frac{1}{2}\left[1 - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin(\gamma B_0 t)\right] = \boxed{\frac{1}{2}\left[1 - \sin\alpha\sin(\gamma B_0 t)\right].}$$

(c)

$$\chi_+^{(z)} = \begin{pmatrix}1\\ 0\end{pmatrix};\quad c_+^{(z)} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \cos\frac{\alpha}{2}e^{i\gamma B_0 t/2};\quad P_+^{(z)}(t) = |c_+^{(z)}|^2 = \boxed{\cos^2\frac{\alpha}{2}.}$$

Problem 4.29

(a)

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}} \text{ (of course).}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow -i\beta = \pm\alpha; \quad |\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha|^2 + |\alpha|^2 = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}.$$

$$\boxed{\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}; \quad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.}$$

(b)

$$c_+ = \left(\chi_+^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib); \quad \boxed{+\frac{\hbar}{2}, \text{ with probability } \frac{1}{2}|a - ib|^2.}$$

$$c_- = \left(\chi_-^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a + ib); \quad \boxed{-\frac{\hbar}{2}, \text{ with probability } \frac{1}{2}|a + ib|^2.}$$

$$\begin{aligned} P_+ + P_- &= \frac{1}{2}\left[(a^* + ib^*)(a - ib) + (a^* - ib^*)(a + ib)\right] \\ &= \frac{1}{2}\left[|a|^2 - ia^*b + iab^* + |b|^2 + |a|^2 + ia^*b - iab^* + |b|^2\right] = |a|^2 + |b|^2 = 1. \checkmark \end{aligned}$$