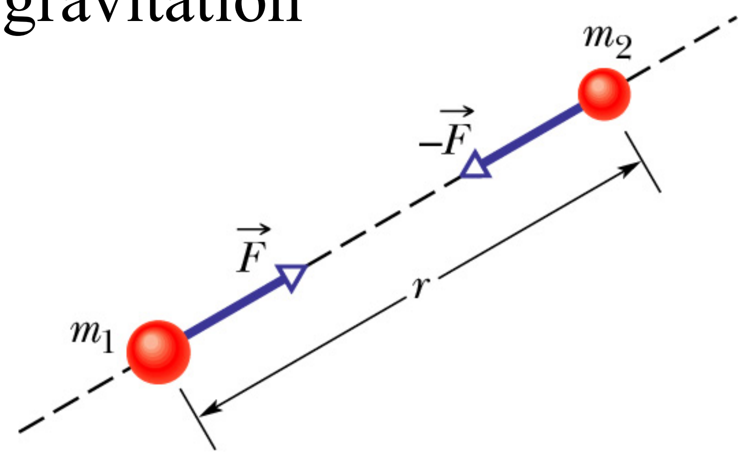
Chapter 13: Gravitation

The force that makes an apple fall is the same force that holds moon in orbit.

Newton's law of gravitation: Every particle attracts any other particle with a gravitation force given by:

$$\vec{F} = G\frac{m_1 m_2}{r^2}(-\hat{r})$$



G: gravitation constant, $G = 6.67 \times 10^{-11} N{\cdot}m^2/kg^2$

The minus sign means this force is always **attractive**.

$$\vec{F} = G\frac{m_1 m_2}{r^2}(-\hat{r})$$

This force depends on the masses *and* the distance squared between them.

Between the earth and a 60 kg person standing on the earth's surface: $F = 588$ N
If the person moved to twice the earth's radius, the force will now be divided by 2^2 (or 4). $F_{2R} = 588N/4 = 147N$

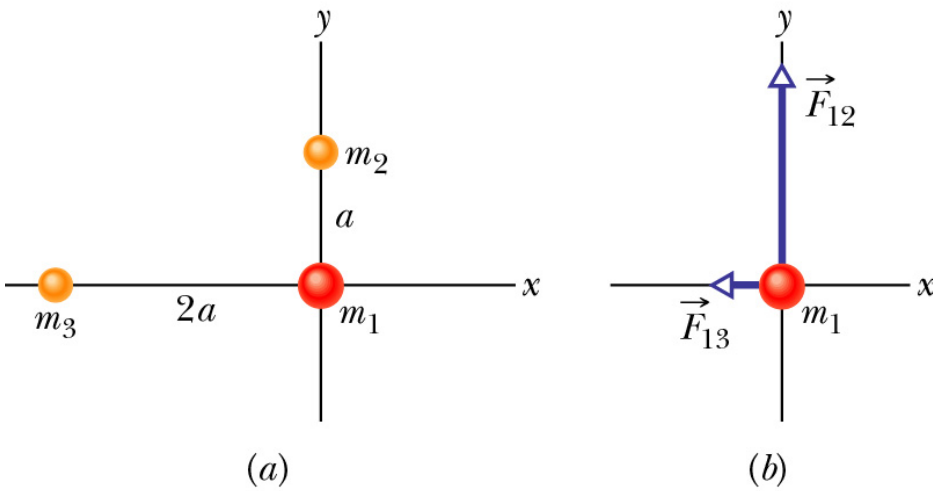
Gravitational force between two 60 kg persons standing 1 m apart: $F = 2.4 \times 10^{-7}$ N

- The principle of superposition: net effect is the sum of the individual effects

$$\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n} = \sum_{i=2}^{n} \vec{F}_{1i}$$

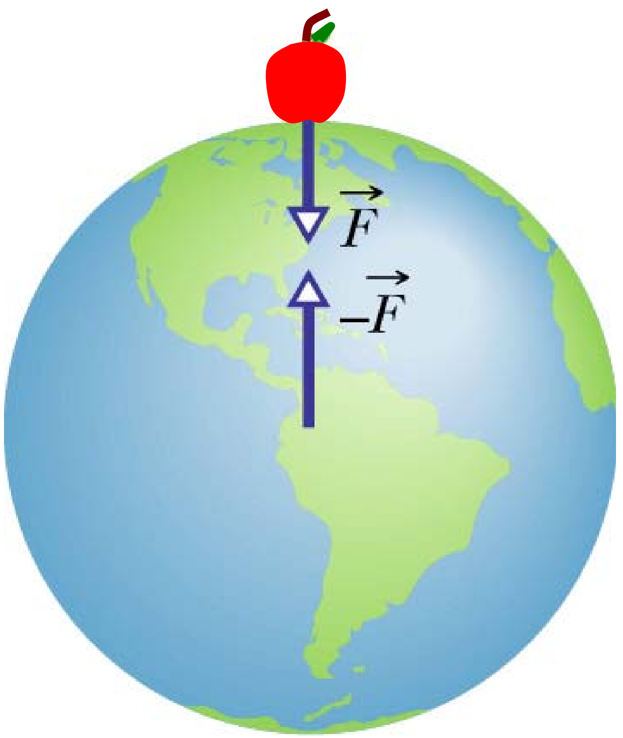
Sample 13-1: What is the net gravitational force F_1 that act on particle 1due to the other two particles?

$$\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13}$$

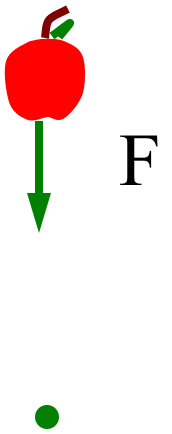


Shell Theorem

Shell theorem: a uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center



is the same as



Gravitation near Earth's Surface

- Assume the Earth is a uniform sphere of mass M , the gravitation force on a particle of mass m located outside earth a distance r from Earth's center is

$$F_g = G\frac{Mm}{r^2} \qquad \text{since } F = m\,a \qquad a_g = G\frac{M}{r^2}$$

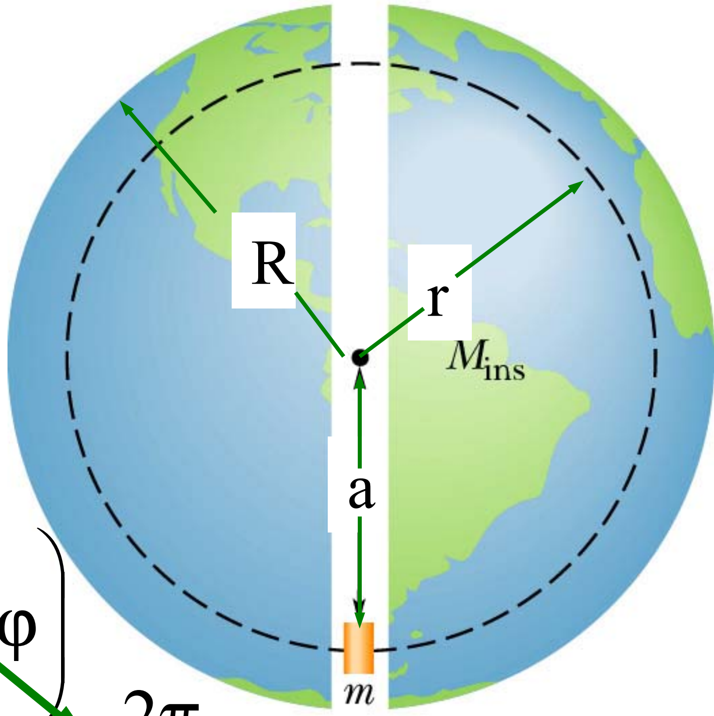
  – a_g varies with attitude,
    Near earth's surface: $a_g = 9.801\ m/s^2$

    at an altitude = 35700 km, $a_g = 0.225\ m/s^2$

Gravitation inside earth

Consider a uniform sphere of matter. We want to find the force on m a radial distance a from the center

Opposing forces cancel for masses r > a. Net force on m comes from the mass that is inside r < a.



$$|F| = G\left(\frac{M_{inside}\,m}{r^2}\right)$$

$$\text{where } M_{inside} = \left(\int_0^a \rho\, r^2 dr\right)\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\varphi\right)$$

2

2π

For a uniform density r, the integral reduces to r times the volume inside a.

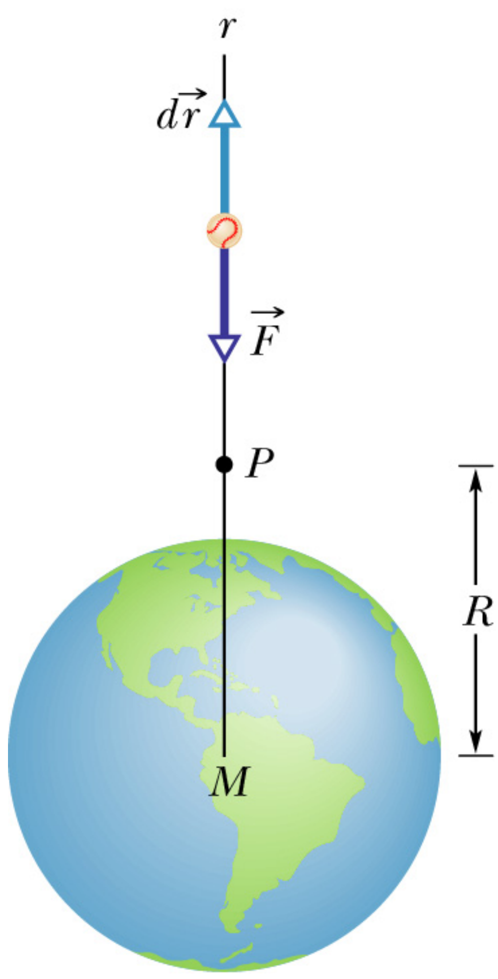
$$M_{inside} = \rho\int_0^a r^2 dr\,(2)(2\pi) = \frac{4}{3}\rho\pi a^3 = M_{total}\left(\frac{a^3}{R^3}\right)$$

Gravitational Potential Energy

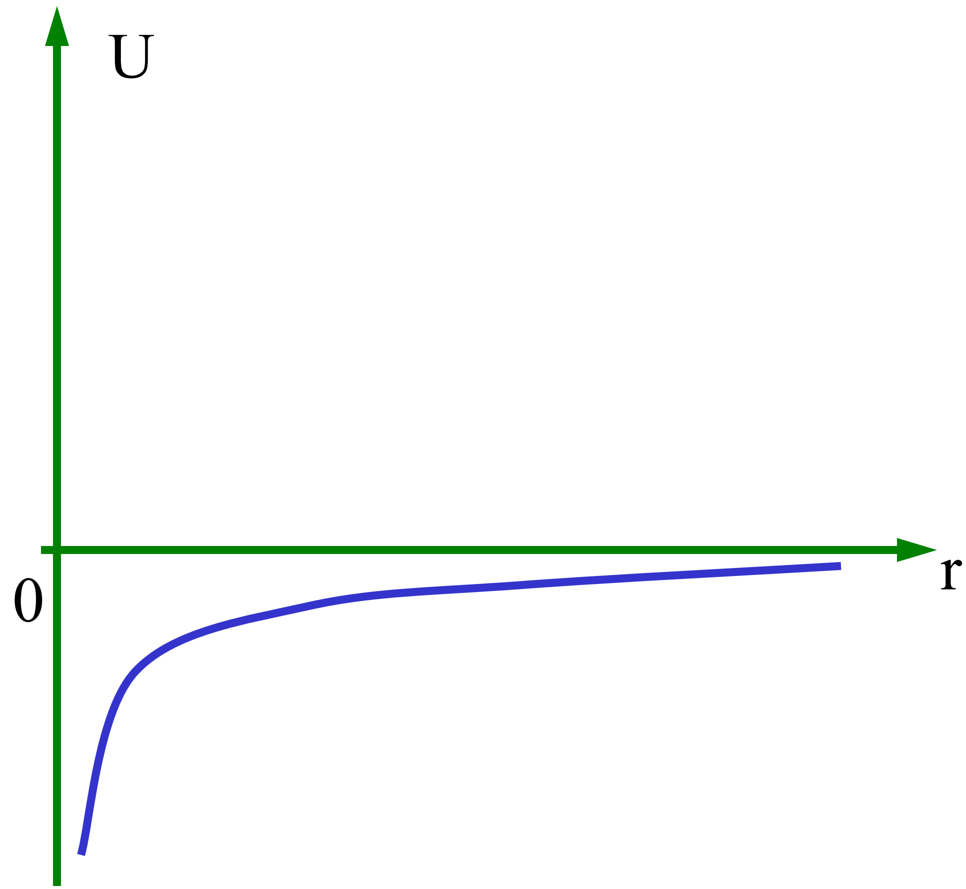
- Gravitational potential energy of a system of two particles M and m:

$$U(r) = -W = -\int \vec{F}(r) \cdot d\vec{r} = -\int F(r) dr \cos 180^o$$

$$= \int F(r) dr = \int \frac{GMm}{r^2} dr = -\frac{GMm}{r} + \text{constant}$$



$$U(r) = -\frac{GMm}{r} + \text{constant}$$

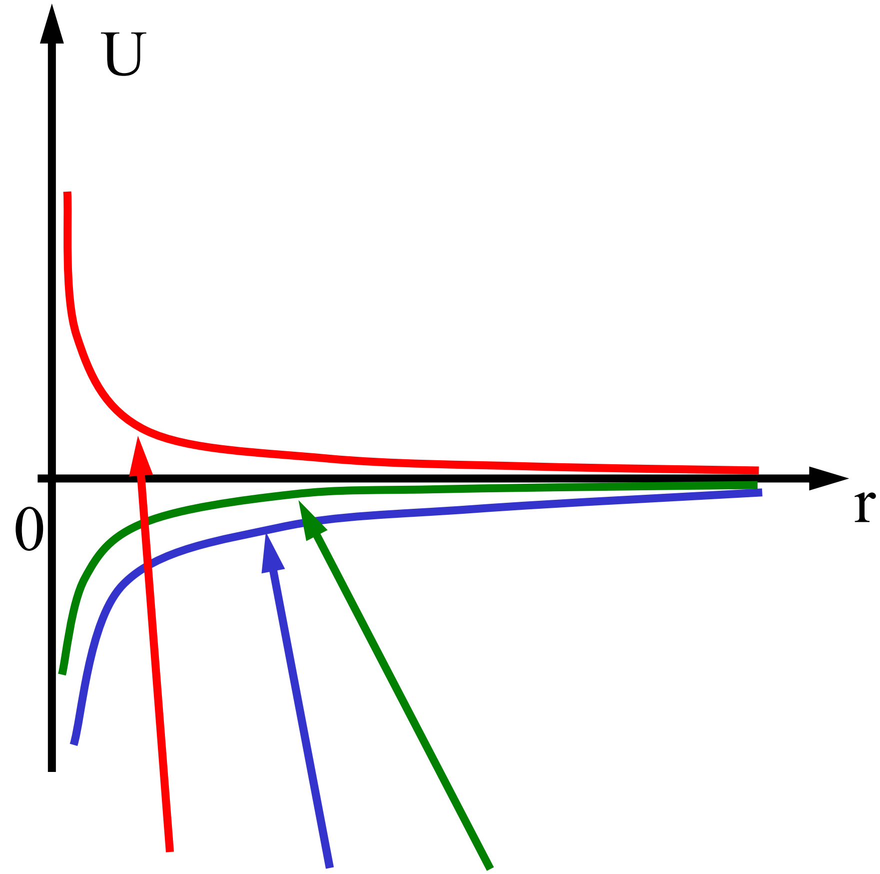


Let $U(r) = 0$ when $r = \infty$ then constant $= 0$

so we have $U(r) = -G\frac{Mm}{r}$

for any finite value of r, U is negative

Escape speed: the minimum initial speed v for a projectile (e.g. rocket) to keep moving upward forever, i.e., $v_{r \to \infty} > 0$



From energy conservation:

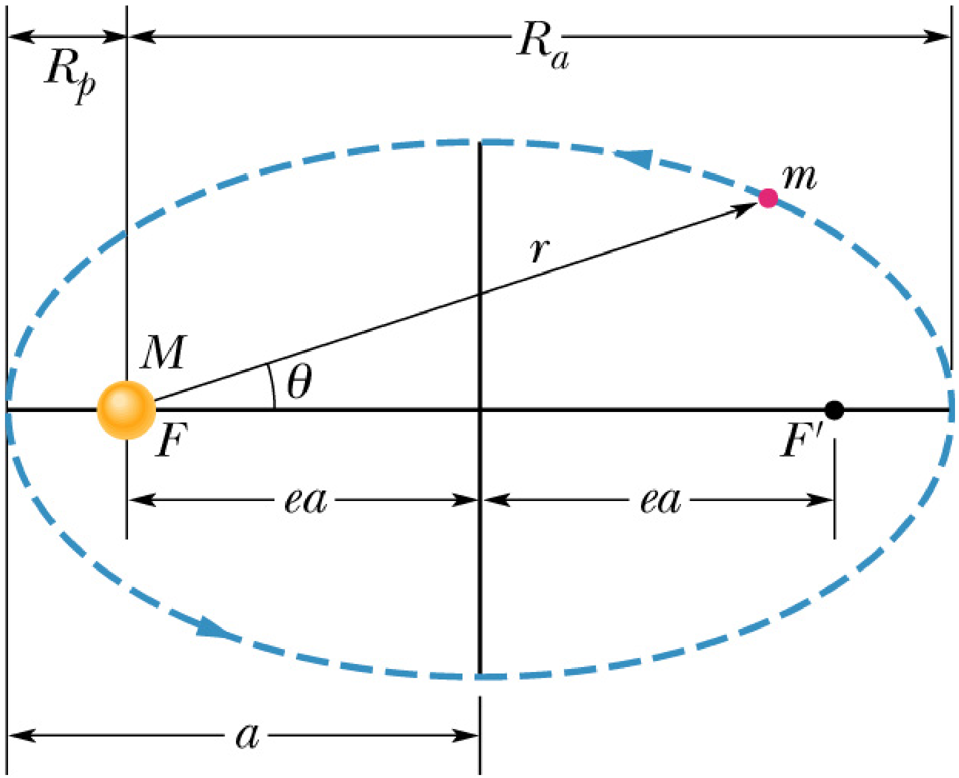
$$K_i + U_i = \tfrac{1}{2} mv^2 + (-GMm/R) = K_f + U_f = E_{tot} >= 0$$

This yields: $v = \sqrt{\dfrac{2GM}{R}}$

Earth: $M = 5.98\text{x}10^{24}$kg, $R = 6.37\text{x}10^{6}$m, v =11.2 km/s

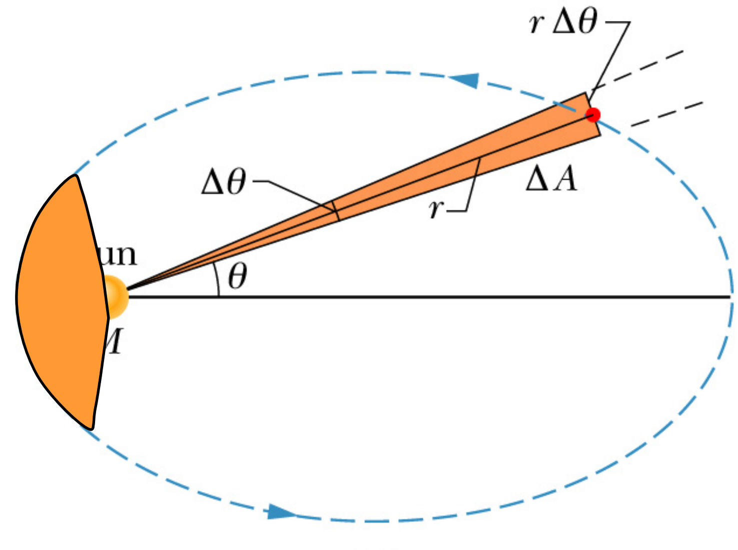
Planets and Satellites: Kepler's laws

The law of orbits: All planets move in elliptical orbits, with the Sun at one focus.



The law of areas: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal times; that is, the rate dA/dt at which it sweeps out area A is constant.

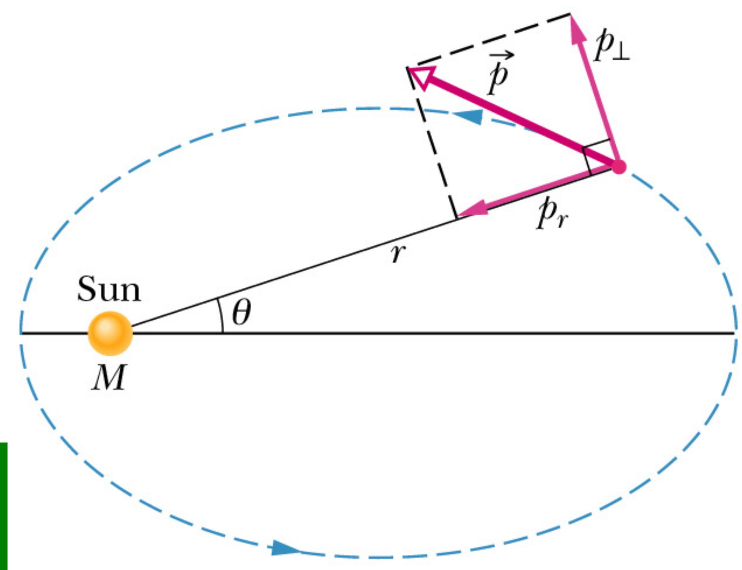
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$$



(a)

$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow$$

$$L = r\,p_{\perp} = (r)(mv_{\perp}) = mr^2\omega$$



(b)

Angular momentum is conserved

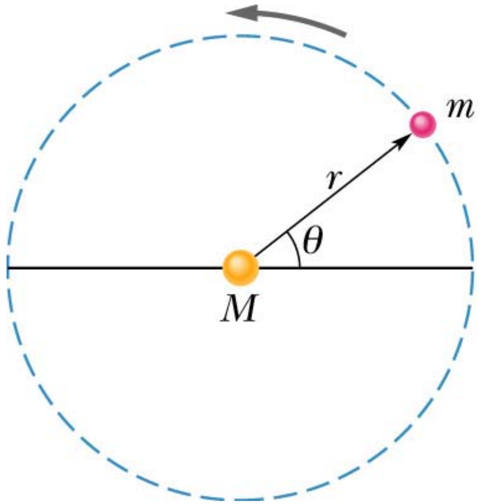
$$\boxed{\frac{dA}{dt} = \frac{L}{2m}}$$

Kepler's Laws

The law of periods: the square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

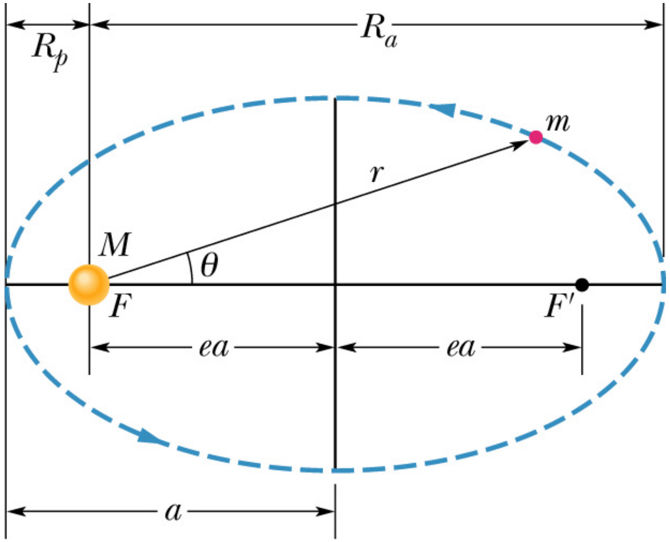
Circular orbit $e = 0$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$



Kepler's Laws

The law of periods: the square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.



Elliptical orbit $e > 0$

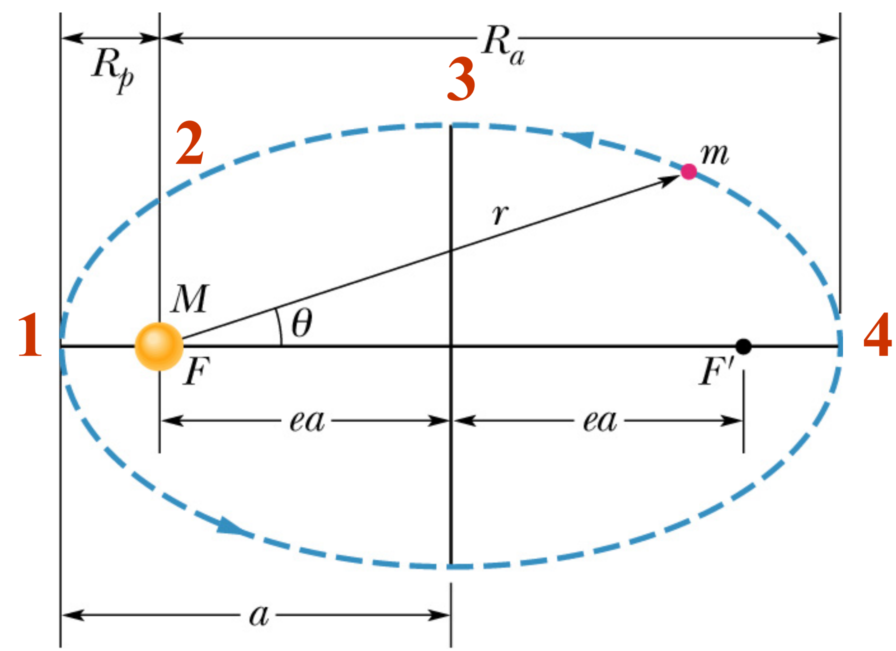
$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

a is the major axis

A Quiz

At which point is m moving the fastest?

1) 1 2) 2 3) 3 4) 4
5) always moves at the same speed
6) some other point on the orbit



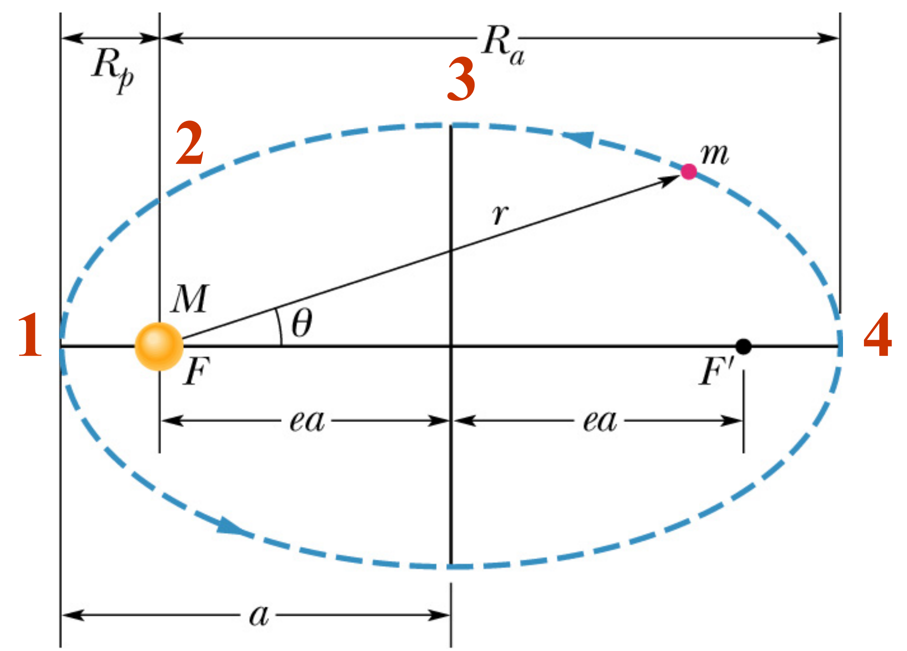
Daily Quiz, March 18, 2004

Reason: m sweeps equal areas in equal times.

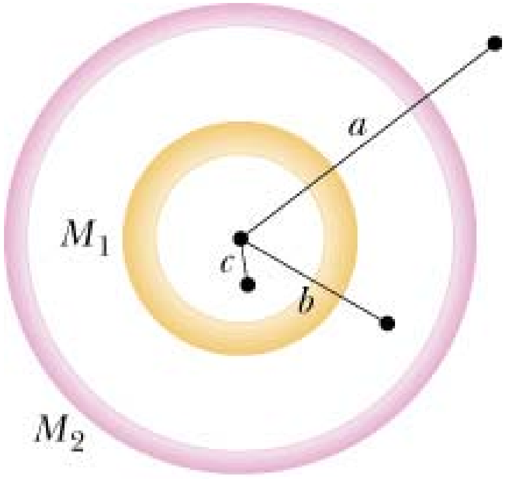
Another way of looking at it: U(r) is most negative at 1, so K must be greatest there to keep E constant.

At which point is m moving the fastest?

1) 1 2) 2 3) 3 4) 4
5) always moves at the same speed
6) some other point on the orbit



Problem 13-20



Two concentric spheres M_1 and M_2. Find F at radii a, b, and c.

$$F_g = G\frac{Mm}{r^2}$$

20. (a) What contributes to the GmM/r^2 force on m is the (spherically distributed) mass M contained within r (where r is measured from the center of M). At point A we see that $M_1 + M_2$ is at a smaller radius than $r = a$ and thus contributes to the force:

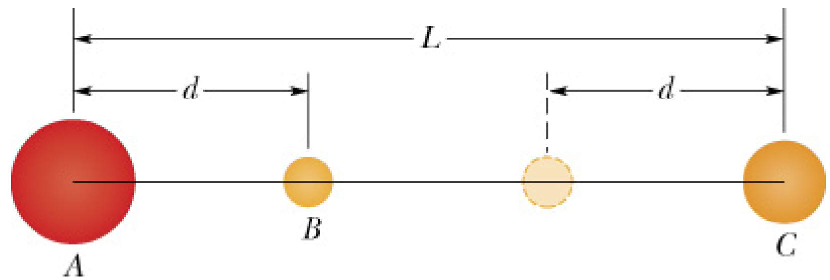
$$\left|F_{\text{on } m}\right| = \frac{G(M_1 + M_2)m}{a^2}.$$

(b) In the case $r = b$, only M_1 is contained within that radius, so the force on m becomes GM_1m/b^2.

(c) If the particle is at C, then no other mass is at smaller radius and the gravitational force on it is zero.

Problem 13-31

Three masses. Move B from near A to near C. Find work done by a) you, b) by gravity.



31. (a) The work done by you in moving the sphere of mass m_B equals the change in the potential energy of the three-sphere system. The initial potential energy is

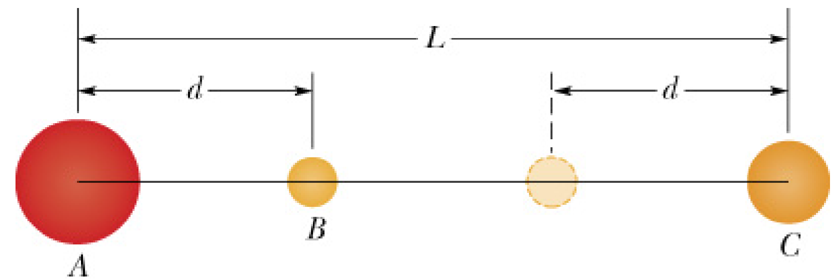
$$U_i = -\frac{Gm_A m_B}{d} - \frac{Gm_A m_C}{L} - \frac{Gm_B m_C}{L-d}$$

and the final potential energy is

$$U_f = -\frac{Gm_A m_B}{L-d} - \frac{GmAm_C}{L} - \frac{Gm_B m_C}{d}.$$

Problem 13-31

Three masses. Move B from near A to near C. Find work done by a) you, b) by gravity.



The work done is

$$W = U_f - U_i = Gm_B\left(m_A\left(\frac{1}{d} - \frac{1}{L-d}\right) + m_C\left(\frac{1}{L-d} - \frac{1}{d}\right)\right)$$

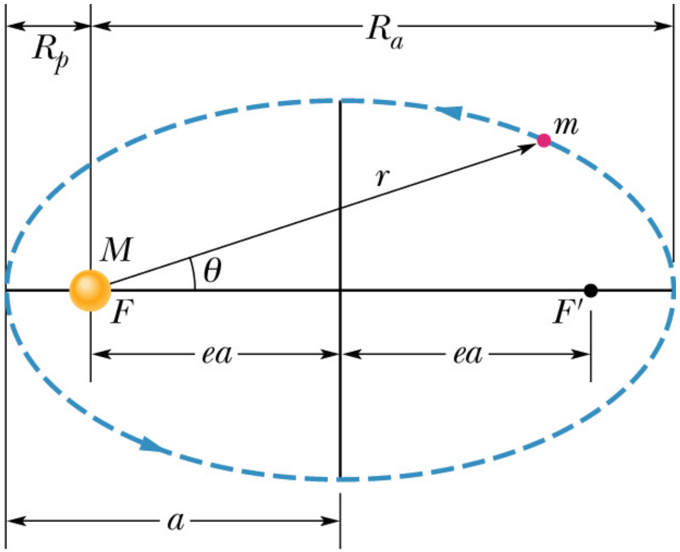
$$= (6.67\times10^{-11}\ \text{m}^3/\text{s}^2\cdot\text{kg})(0.010\ \text{kg})\left[(0.080\ \text{kg})\left(\frac{1}{0.040\ \text{m}} - \frac{1}{0.080\ \text{m}}\right) + (0.020\ \text{kg})\left(\frac{1}{0.080\ \text{m}} - \frac{1}{0.040\ \text{m}}\right)\right]$$

$$= \ +5.0\times10^{-13}\ \text{J}.$$

(b) The work done by the force of gravity is $-(U_f - U_i) = -5.0\times10^{-13}$ J.

Problem 13-44

Find distance between the foci of the Earth's orbit.



44. (a) The distance from the center of an ellipse to a focus is *ae* where *a* is the semimajor axis and *e* is the eccentricity. Thus, the separation of the foci (in the case of Earth's orbit) is

$$2ae = 2\left(1.50\times10^{11}\ \text{m}\right)\left(0.0167\right) = 5.01\times10^{9}\ \text{m}.$$

(b) To express this in terms of solar radii (see Appendix C), we set up a ratio:

$$\frac{5.01\times10^{9}\ \text{m}}{6.96\times10^{8}\ \text{m}} = 7.20.$$

Problem 13-46

Find distance for geosynchronous orbit.

http://science.nasa.gov/Realtime/Jtrack/3d/JTrack3D.html

46. To "hover" above Earth ($M_E = 5.98 \times 10^{24}$ kg) means that it has a period of 24 hours (86400 s). By Kepler's law of periods,

$$(86400)^2 = \left(\frac{4\pi^2}{GM_E}\right) r^3 \Rightarrow r = 4.225 \times 10^7 \text{ m}.$$

Its altitude is therefore $r - R_E$ (where $R_E = 6.37 \times 10^6$ m) which yields 3.58×10^7 m.

Satellites and Orbits

Potential energy $$U(r) = -G\frac{Mm}{r}$$

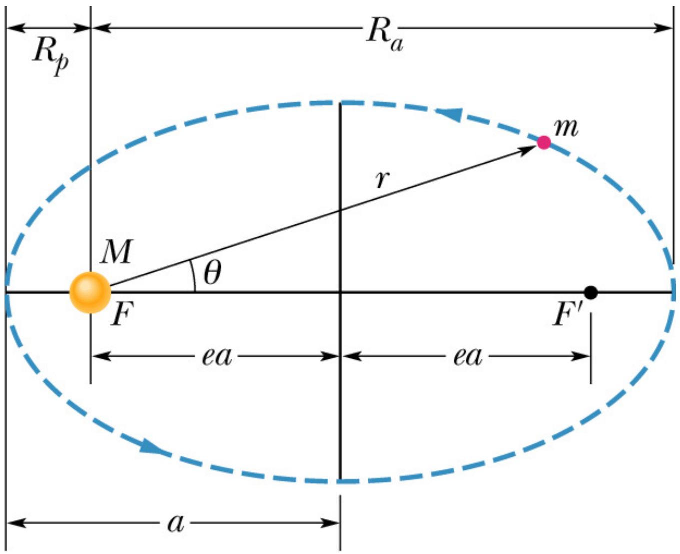
Centripetal force $$F_c = m\frac{v^2}{r} = G\frac{Mm}{r^2}$$

Kinetic energy $$K = \frac{1}{2}mv^2 = G\frac{Mm}{2r} = -\frac{1}{2}U$$

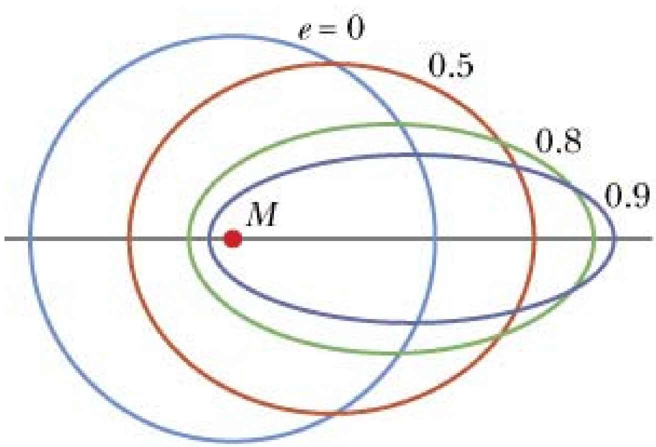
Total energy $$E = K + U = \left(-\frac{1}{2}U\right) + U = -G\frac{Mm}{2r}$$

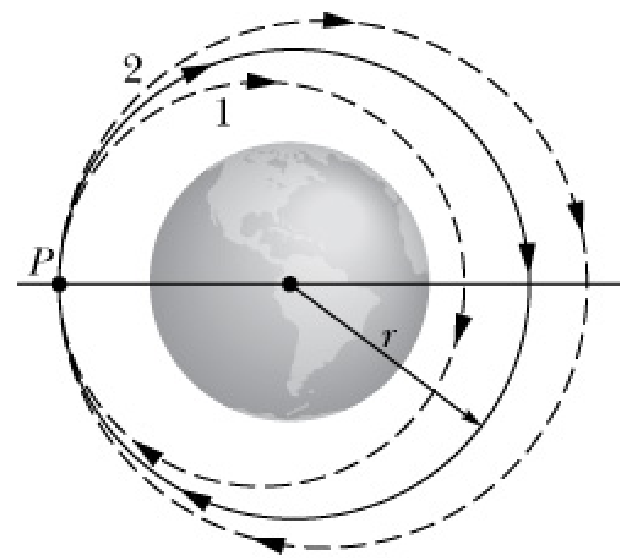
elliptical a (replaces r)

Elliptical Orbits



Total energy $E = -G\dfrac{Mm}{2a}$



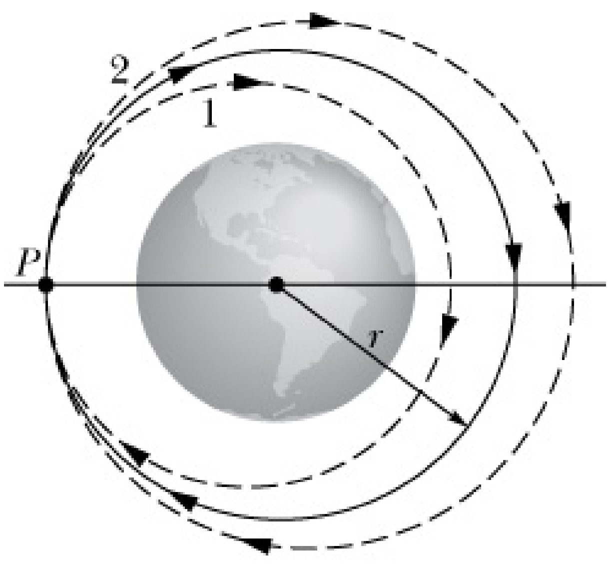


A Quiz

All three orbits intersect at P. Which path has the greater total energy?

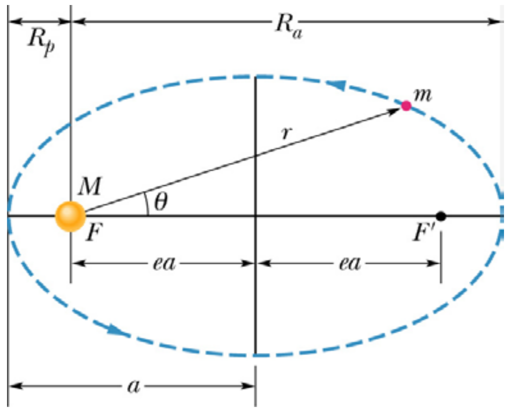
1) 1   2) 2   3) 3

4) all have the same total energy

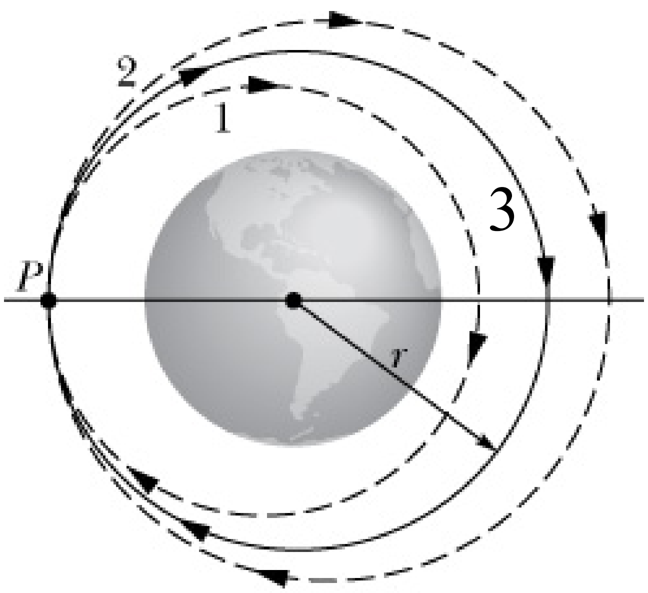


A Quiz

Total Energy $E = -G\dfrac{Mm}{2a}$



$a_1 < a_3 < a_2$ => E_2 is least negative.



All three orbits intersect at P. Which path has the greater total energy?

1) 1 2) 2 3) 3

4) all have the same total energy