Chapter 3: Vectors

- To describe motions in 2- or 3-dimensions, we need vectors

- A vector quantity has both a magnitude and a direction. e.g., acceleration, velocity, displacement, force, torque, and momentum.

- A scalar quantity does not involve a (spatial) direction. e.g., charge, mass, time, temperature, energy, etc.
Vector addition

• Graphical method
e. g. two vectors \( \mathbf{a}, \mathbf{b} \)

\[
\mathbf{a} + \mathbf{b} =
\]

\[
\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) =
\]
Check point 3.1: The magnitude of displacements \( \mathbf{a} \) and \( \mathbf{b} \) are 3 m and 4 m, respectively, and \( \mathbf{c} = \mathbf{a} + \mathbf{b} \). Considering various orientations of \( \mathbf{a} \) and \( \mathbf{b} \), what is (a) the maximum possible magnitude for \( \mathbf{c} \) (b) the minimum possible magnitude for \( \mathbf{c} \)
Vector Addition Property

\[ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \]
(commutative)

\[(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})\]
(associative)
Vector in a coordinate system

Magnitude-angle notation: 

\[ a_x = a \cos \theta \]
\[ a_y = a \sin \theta \]

a: magnitude

\( \theta \): relative to +x direction, counter-clockwise is positive: “clock is negative”
Components of Vectors

\[ a_x = a \cos \theta \]
\[ a_y = a \sin \theta \]

- Component notation vs magnitude-angle notation

\[ a = \sqrt{a_x^2 + a_y^2} \]
\[ \tan \theta = \frac{a_y}{a_x} \]
Unit Vectors

- Has a magnitude of 1 and points in a particular direction

- $i$, $j$, $k$, unit vectors in the positive $x$, $y$, $z$ direction, follow right-handed coordinate system

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} \]
Add vectors by components

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \]
\[ \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \]
\[ \vec{r} = \vec{a} + \vec{b} = (a_x + b_x) \hat{i} + (a_y + b_y) \hat{j} + (a_z + b_z) \hat{k} \]
Example: In the figure below,
a) What are the signs of the x components of $d_1$ and $d_2$?
b) What are signs of the y components of $d_1$ and $d_2$?
c) What are the signs of the x and y components of $d_1 + d_2$?
d) What is the final vector $d_1 + d_2$?
In which quadrant would $\mathbf{a} + \mathbf{b}$ be located if

$\mathbf{a} = 3.0 \mathbf{i} - 4.0 \mathbf{j}$ and

$\mathbf{b} = -2.0 \mathbf{i} + 2.0 \mathbf{j}$?

1) 2) 3) 4) 0) none of the above
A Quiz

\[
\vec{a} = 3.0\hat{i} - 4.0\hat{j} + 0.0\hat{k}
\]
\[
\vec{b} = -2.0\hat{i} + 2.0\hat{j} + 0.0\hat{k}
\]
\[
\vec{r} = \vec{a} + \vec{b} = (3.0 - 2.0)\hat{i} + (-4.0 + 2.0)\hat{j} + (0.0 + 0.0)\hat{k}
\]
\[
\vec{r} = (1.0)\hat{i} + (-2.0)\hat{j} + (0.0)\hat{k}
\]

In which quadrant would \( \vec{a} + \vec{b} \) be located if \( \vec{a} = 3.0 \, \hat{i} - 4.0 \, \hat{j} \) and \( \vec{b} = -2.0 \, \hat{i} + 2.0 \, \hat{j} \)?
Multiplication of Vectors

• **Multiply a vector by a scalar:** \( \mathbf{b} = s \mathbf{a} \)
  – Magnitude of \( \mathbf{b} \): \( s \) times the magnitude of \( \mathbf{a} \)
  – Direction of \( \mathbf{b} \): same as \( \mathbf{a} \) if \( s > 0 \),
    opposite of \( \mathbf{a} \) if \( s < 0 \)

• **Multiply a vector by a vector**
  – **Scalar product** (tells you the projection of \( \mathbf{a} \) onto \( \mathbf{b} \).)
    • results in a scalar
  – **Vector product** (tells you the area subtended by \( \mathbf{a} \) and \( \mathbf{b} \).)
    • results in another vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \).
Scalar product

• Scalar product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \)

\[
\mathbf{a} \cdot \mathbf{b} = |a| |b| \cos \phi
\]

\( a, b \): magnitude of \( \mathbf{a}, \mathbf{b} \)

\( \phi \): angle between the directions of \( \mathbf{a} \) and \( \mathbf{b} \)
Scalar product

\[ \mathbf{a} \cdot \mathbf{b} = a \, b \cos \phi \]

If \( a \) and \( b \) parallel, \( \phi = 0^\circ \)

\[ \Rightarrow \mathbf{a} \cdot \mathbf{b} = a \, b \cos 0^\circ = a \, b \]

If \( a \) and \( b \) perpendicular, \( \phi = 90^\circ \)

\[ \Rightarrow \mathbf{a} \cdot \mathbf{b} = a \, b \cos 90^\circ = 0 \]

\[ \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0 \]
Scalar Product

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \]

\[ \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \]

\[ \vec{a} \cdot \vec{b} = (a_x b_x) + (a_y b_y) + (a_z b_z) \]
Check point 3-4: Vectors \( \mathbf{C} \) and \( \mathbf{D} \) have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of \( \mathbf{C} \) and \( \mathbf{D} \) if \( \mathbf{C} \cdot \mathbf{D} \) is

(a) zero

(b) 12 units

(c) –12 units
What is the scalar product between
\( \mathbf{a} = 3.0 \, \mathbf{i} - 4.0 \, \mathbf{j} \) and
\( \mathbf{b} = -2.0 \, \mathbf{i} + 2.0 \, \mathbf{j} \)?

\[
\begin{align*}
\mathbf{\hat{a}} &= 3.0 \hat{\mathbf{i}} - 4.0 \hat{\mathbf{j}} \\
\mathbf{\hat{b}} &= -2.0 \hat{\mathbf{i}} + 2.0 \hat{\mathbf{j}} \\
\mathbf{\hat{a}} \cdot \mathbf{\hat{b}} &= (3.0)(-2.0) + (-4.0)(2.0) = -14.0
\end{align*}
\]
What is the angle between \( \mathbf{a} = 3.0 \mathbf{i} - 4.0 \mathbf{j} \) and \( \mathbf{b} = -2.0 \mathbf{i} + 2.0 \mathbf{j} \)?

\[
\mathbf{a} \cdot \mathbf{b} = a \cdot b \cos \phi
\]

\[
a = 5 \\
b = 2.828
\]

\[
\mathbf{a} \cdot \mathbf{b} = (3.0)(-2.0) + (-4.0)(2.0) + (0.0)(0.0) = -6.0 - 8.0 = -14.0
\]

\[
\cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{(a \cdot b)} = \frac{-14.0}{14.142} = -0.990
\]

\[
\phi = 171.9^\circ
\]

What is the angle between \( \mathbf{a} = 3.0 \mathbf{i} - 4.0 \mathbf{j} \) and \( \mathbf{b} = -2.0 \mathbf{i} + 2.0 \mathbf{j} \)?
Vector Product

- Vector product of two vectors $a$ and $b$ produce a third vector $c$ whose magnitude is
  
  $$c = a \cdot b \sin \phi$$

  whose direction follow the right hand rule

Note: $a \times b = - (b \times a)$
Vector product

• \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \implies \mathbf{c} = \mathbf{a} \mathbf{b} \sin \phi \)

• if \( \mathbf{a} \) and \( \mathbf{b} \) parallel, \( \phi = 0^\circ \)
  \( \implies \mathbf{a} \times \mathbf{b} = 0 \)
  if \( \mathbf{a} \) and \( \mathbf{b} \) perpendicular, \( \phi = 90^\circ \)
  \( \implies \mathbf{c} = \mathbf{a} \mathbf{b} \)

\[
\begin{align*}
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\
\mathbf{i} \times \mathbf{j} &= \mathbf{j} \times \mathbf{k} = \mathbf{k} \times \mathbf{i} = 1 \\
\mathbf{j} \times \mathbf{i} &= \mathbf{k} \times \mathbf{j} = \mathbf{i} \times \mathbf{k} = -1
\end{align*}
\]
Check point 3-5: Vectors \( \mathbf{C} \) and \( \mathbf{D} \) have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of \( \mathbf{C} \) and \( \mathbf{D} \) if the magnitude of the vector products \( \mathbf{C} \times \mathbf{D} \) is

(a) Zero

(b) 12 units