Chapter 11: Rolling, Torque, and Angular Momentum

• For an object rolling smoothly, the motion of the center of mass is pure translational.

\[ s = \theta R \quad v_{\text{com}} = \frac{ds}{dt} = \frac{d(\theta R)}{dt} = \omega R \]

\[ v_{\text{com}} = \omega R \]
• Rolling viewed as a combination of pure rotation and pure translation

\[ \vtop = (\omega)(2R) = 2 v_{com} \]

• Rolling viewed as pure rotation

\[ v_{top} = (\omega)(2R) = 2 v_{com} \]

• Different views, same conclusion
Rotational inertia involves not only the mass but also the distribution of mass for continuous masses.

Calculating the rotational inertia: $I = \int r^2 \, dm$

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<td>Hoop about central axis</td>
<td>$I = MR^2$</td>
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<td><img src="image2.png" alt="Image" /></td>
<td>Annular cylinder (or ring) about central axis</td>
<td>$I = \frac{1}{2} M (R_1^2 + R_2^2)$</td>
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<td><img src="image3.png" alt="Image" /></td>
<td>Solid cylinder (or disk) about central axis</td>
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<td>Solid cylinder (or disk) about central diameter</td>
<td>$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$</td>
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<td>Thin rod about axis perpendicular to length</td>
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<td>Solid sphere about any diameter</td>
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<tr>
<td><img src="image8.png" alt="Image" /></td>
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<td><img src="image9.png" alt="Image" /></td>
<td>Slab about perpendicular axis through center</td>
<td>$I = \frac{1}{12} M(a^2 + b^2)$</td>
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The Kinetic Energy of Rolling

View the rolling as pure rotation around P, the kinetic energy

\[ K = \frac{1}{2} I_P \omega^2 \]

parallel axis theorem: \( I_p = I_{com} + MR^2 \)

so \( K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} MR^2 \omega^2 \)

since \( v_{com} = \omega R \)

\[ K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M(v_{com})^2 \]

\( \frac{1}{2} I_{com} \omega^2 \): due to the object’s rotation about its center of mass

\( \frac{1}{2} M(v_{com})^2 \): due to the translational motion of its center of mass
Sample Problem: A uniform solid cylindrical disk, of mass $M = 1.4 \text{ kg}$ and radius $R = 8.5 \text{ cm}$, rolls smoothly across a horizontal table at a speed of $15 \text{ cm/s}$. What is its kinetic energy $K$?

$v_{c.m.} = 0.15 \text{ m/s}$

$I_{\text{disk}} = \frac{1}{2}MR^2 = (0.5)(1.4\text{ kg})(0.085\text{ m})^2 = 5.058 \times 10^{-3} \text{ kg m}^2$

$\omega = \frac{v}{R} = \frac{0.15 \text{ m/s}}{0.085 \text{ m}} = 1.765 \text{ rad/s}$

$K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}Mv_{c.m.}^2 + \frac{1}{2}I\omega^2$

$K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}(1.4)(0.15)^2 + \frac{1}{2}(5.058 \times 10^{-3})(1.765)^2$

$= 15.75 \times 10^{-3} + 7.878 \times 10^{-3} = 23.63 \times 10^{-3} \text{ J}$
Vector Product (Review)

- Vector product of vectors \( \mathbf{a} \) and \( \mathbf{b} \) produce a third vector \( \mathbf{c} \) whose magnitude is
  \[
  c = |\mathbf{a}| |\mathbf{b}| \sin \phi
  \]
  whose direction follow the right hand rule

if \( \mathbf{a} \) and \( \mathbf{b} \) parallel \( \Rightarrow \) \( \mathbf{a} \times \mathbf{b} = 0 \)
if \( \mathbf{a} \) and \( \mathbf{b} \) perpendicular \( \Rightarrow \) \( \mathbf{a} \times \mathbf{b} = \mathbf{ab} \)
\( \mathbf{i} \times \mathbf{i} = 0 \quad \mathbf{i} \times \mathbf{j} = 1 \)
Note: \( \mathbf{a} \times \mathbf{b} = - (\mathbf{b} \times \mathbf{a}) \)
Torque revisited

- For a fixed axis rotation, torque $\tau = r F \sin \theta$
- Expand the definition to apply to a particle that moves along any path relative to a fixed point.

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

- direction: right-hand rule
- magnitude: $\tau = rF \sin \Phi = rF_{\perp} = r_{\perp}F$
Angular momentum

- Angular momentum with respect to point O for a particle of mass $m$ and linear momentum $p$ is defined as

$$\vec{\ell} = \vec{r} \times \vec{p} = m (\vec{r} \times \vec{v})$$

direction: right-hand rule

magnitude:

$$\ell = r \ p \ \sin \phi = r \ m v \ \sin \phi$$

- Compare to the linear case $\dot{\vec{p}} = m \dot{\vec{v}}$
Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

Which of the particles has the greatest magnitude angular momentum?
Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

\[ \ell = r p \sin \phi = r mv \sin \phi \quad \phi = 90^\circ \text{ only for 1 and 2, but } r_1 = 2r_2. \]

Which of the particles has the greatest magnitude angular momentum?

1) 1  
2) 2  
3) 3  
4) 4  
5) 5  
6) all have the same $l$
Newton’s Second Law in Angular Form

\[ \vec{\tau}_{net} = \frac{d\vec{l}}{dt} \]

- \( \vec{\tau}_{net} \) : the vector sum of all the torques acting on the object
- Comparing to the linear case: \( \vec{F}_{net} = \frac{d\vec{p}}{dt} \)

- Newton’s 2\textsuperscript{nd} law for a system of particles

\[ \vec{\tau}_{net} = \frac{d\vec{L}_{total}}{dt} \quad \vec{L}_{total} = \sum_{i=1}^{n} l_i \]

- Net external torque equals to the time rate change of the system’s total angular momentum
Torque and Angular Momentum

\[ \tau_{\text{net}} = \frac{d\vec{l}}{dt} = \frac{d(\vec{r} \times m\vec{v})}{dt} = \vec{r} \times \left( m \frac{d\vec{v}}{dt} \right) = \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F} \]

**Torque** is the time rate of change of angular momentum.
A Quiz

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

Which of the particles has the smallest magnitude angular momentum?

1) 1      2) 2      3) 3     4) 4     5) 5     6) all have the same $l$
A Quiz

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

\[ l = r p \sin \phi = r mv \sin \phi \]

\[ \phi = 0^o \text{ for 5. } \Rightarrow l = 0 \]

Which of the particles has the smallest magnitude angular momentum?

1) 1  2) 2  3) 3  4) 4  [5) 5]  6) all have the same $l$
The angular momentum of a rigid body rotating about a fixed axis

• Consider a simple case, a mass $m$ rotating about a fixed axis $z$:
  \[ l = r \cdot m \cdot v \cdot \sin 90^\circ = r \cdot m \cdot r \cdot \omega = m r^2 \omega = I \omega \]

• In general, the angular momentum of rigid body rotating about a fixed axis is
  \[ L = I \omega \]

  $L$ : angular momentum along the rotation axis
  $I$ : moment of inertia about the same axis
Conservation of Angular Momentum

• If the net external torque acting on a system is zero, the angular momentum of the system is conserved.

\[ \bar{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad \text{\( \bar{\tau}_{\text{net}} = 0 \)} \quad \text{then} \quad \vec{L} = \text{const} \]

• If \( \tau_{\text{net}, z} = 0 \) then \( L_{i, z} = L_{f, z} \quad \vec{L}_i = \vec{L}_f \)

• For a rigid body rotating around a fixed axis, ( \( L = I \omega \) ) the conservation of angular momentum can be written as

\[ I_i \omega_i = I_f \omega_f \]
Some examples involving conservation of angular momentum

- The spinning volunteer

\[ I_i \omega_i = I_f \omega_f \]
Angular momentum is conserved

$L_i$ is in the spinning wheel

Now exert a torque to flip its rotation.
$L_f, \text{wheel} = -L_i$.

Conservation of Angular momentum means that the person must now acquire an angular momentum.
$L_f, \text{person} = +2L_i$

so that $L_f = L_f, \text{person} + L_f, \text{wheel} = +2L_i - L_i = L_i$. 
Problem 11-66

Ring of $R_1 (=R_2/2)$ and $R_2 (=0.8 \text{m})$, Mass $m_2 = 8.00 \text{kg}$.\n$\omega_i = 8.00 \text{ rad/s}$. Cat $m_1 = 2\text{kg}$. Find kinetic energy change when cat walks from outer radius to inner radius.

Initial Momentum

$$L_i = L_{i,\text{cat}} + L_{i,\text{ring}} = m_1 R_2 v_i + I \omega_i$$

$$= m_1 R_2^2 \omega_i + \frac{1}{2} m_2 (R_1^2 + R_2^2) \omega_i$$

$$= m_1 R_2^2 \omega_i \left(1 + \frac{1}{2} \frac{m_2}{m_1} \left(\frac{R_1^2}{R_2^2} + 1\right)\right)$$
Problem 11-66

Ring of $R_1 (=R_2/2)$ and $R_2 (=0.8\text{m})$, Mass $m_2 = 8.00\text{kg}$. 
$\omega_i = 8.00\text{ rad/s}$. Cat $m_1 = 2\text{kg}$. Find kinetic energy change when cat walks from outer radius to inner radius.
Problem 11-66

Ring of $R_1 (=R_2/2)$ and $R_2 (=0.8\text{m})$, Mass $m_2 = 8.00\text{kg}$.
Angular velocity $\omega_i = 8.00\text{ rad/s}$. Cat $m_1 = 2\text{kg}$. Find kinetic energy change when cat walks from outer radius to inner radius.

**Final Momentum**

$$L_f = L_{f,\text{cat}} + L_{f,\text{ring}} = m_1 R_1 v_f + I \omega_f$$

$$= m_1 R_1^2 \omega_f + \frac{1}{2} m_2 (R_1^2 + R_2^2) \omega_f$$

$$= m_1 R_1^2 \omega_f \left(1 + \frac{1}{2} \frac{m_2}{m_1} \left(\frac{R_2^2}{R_1^2} + 1\right)\right)$$
Problem 11-66

Then from $L_f = L_i$ we obtain

$$\frac{\omega_f}{\omega_0} = \frac{R_2^2}{R_1^2} \frac{1 + \frac{1}{2} \frac{m_2}{m_1} \left( \frac{R_1^2}{R_2^2} + 1 \right)}{1 + \frac{1}{2} \frac{m_2}{m_1} \left( 1 + \frac{R_2^2}{R_1^2} \right)} = \left(2.0\right)^2 \frac{1 + 2(0.25 + 1)}{1 + 2(1 + 4)} = 1.273$$

Thus, $\omega_f = 1.273 \omega_0$. Using $\omega_0 = 8.00$ rad/s, we have $\omega_f = 10.2$ rad/s. By substituting $I = L/\omega$ into $K = \frac{1}{2} I \omega^2$, we obtain $K = \frac{1}{2} L \omega$. Since $L_i = L_f$, the kinetic energy ratio becomes

$$\frac{K_f}{K_i} = \frac{\frac{1}{2} L_f \omega_f}{\frac{1}{2} L_i \omega_i} = \frac{\omega_f}{\omega_0} = 1.273.$$ 

which implies $\Delta K = K_f - K_i = 0.273 K_i$. The cat does positive work while walking toward the center of the ring, increasing the total kinetic energy of the system.
Problem 11-66

Ring of \( R_1 (=R_2/2) \) and \( R_2 (=0.8m) \),
Mass \( m_2 = 8.00kg \).
\( \omega_i = 8.00 \) rad/s. Cat \( m_1 = 2\)kg. Find
kinetic energy change when cat walks
from outer radius to inner radius.

Initial Kinetic energy \( K_i \) is:

\[
K_i = \frac{1}{2} \left[ m_1 R_2^2 + \frac{1}{2} m_2 (R_1^2 + R_2^2) \right] \omega_0^2 = \frac{1}{2} m_1 R_2^2 \omega_0^2 \left[ 1 + \frac{1}{2} \frac{m_2}{m_1} \left( \frac{R_1^2}{R_2^2} + 1 \right) \right]
\]

\[
= \frac{1}{2} (2.00 \text{ kg})(0.800 \text{ m})^2 (8.00 \text{ rad/s})^2 [1+(1/2)(4)(0.5^2+1)]
\]

\[
=143.36 \text{ J},
\]

the increase in kinetic energy is \( \Delta K = (0.273)(143.36 \text{ J})=39.1 \text{ J} \).