Chapter 6: Oscillations

Jigsaw Theory 2

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The Eigenvalue Problem

- Non-degenerate eigenvalues
- Let λ_k be the n non-degenerate eigenvalues and $\bar{a}_k$ be the eigenvectors of $\bar{\bar{T}}^{-1}\bar{\bar{V}}$
- □

$$\bar{\bar{T}}^{-1}\bar{\bar{V}}\bar{a}_k = \lambda_k \bar{a}_k$$

- Where

$$\bar{a}_k \equiv \begin{bmatrix} a_{k1} \\ a_{k2} \\ . \\ . \\ a_{k3} \end{bmatrix}$$

Define Unit Vectors

- Let $\hat{e}_k$ be unit vectors such that $e_{ki} = \delta_{ik}$
- In other words the vectors represented by an n x 1 matrix where all of its elements are zero except 1 at the k^th row

$$\hat{e}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \\ . \\ . \\ 0 \end{bmatrix}_{n\times 1}$$

Define A Matrix

- Define

$$\bar{\bar{A}}_{n\times n} = \sum_k \hat{e}_k \cdot \bar{a}_k^T$$

$$A_{ij} = \left[\sum_k \hat{e}_k \cdot \bar{a}_k^T\right]_{ij} = \sum_k e_{ki} a_{kj}$$

$$= \sum_k \delta_{ik} a_{kj} = a_{ij}$$

- ➔ The k[th] column of $\bar{\bar{A}}$s $\bar{a}_k$

Use A in Eigenvalue Problem

- Original Form

$$\bar{\bar{T}}^{-1}\bar{\bar{V}}\bar{a}_k = \lambda_k \bar{a}_k$$

$$\bar{\bar{V}}\bar{a}_k = \lambda_k \bar{\bar{T}}\bar{a}_k$$

- Using $\bar{\bar{A}}$

$$\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}}$$

- Where $\bar{\bar{\lambda}}$ is a n x n matrix which is diagonal
  - →

$$\lambda_{ij} = \lambda_i \delta_{ij}$$

Useful Properties

- Define the Hermitian conjugate by a dagger superscript □such that

  $$\bar{\bar{B}}^{\dagger} \equiv \left(\bar{\bar{B}}^{T}\right)^{*} \equiv \left(\bar{\bar{B}}^{*}\right)^{T}$$

  where $\left(\bar{\bar{B}}^{*}\right)_{ij} \equiv \left(B_{ij}\right)^{*} \equiv B_{ij}^{*}$

- Where * denotes complex conjugation

  $$(a+ib)^{*} = a - ib \ \ or \ \ \left(re^{i\theta}\right)^{*} = re^{-i\theta}$$

  where r is real

- $\bar{\bar{V}}$ and $\bar{\bar{T}}$ are real and $\bar{\bar{V}}^{T} = \bar{\bar{V}} \ \& \ \bar{\bar{T}}^{T} = \bar{\bar{T}}$

  → $\bar{\bar{V}}^{\dagger} = \bar{\bar{V}} \ \& \ \bar{\bar{T}}^{\dagger} = \bar{\bar{T}}$

Applying to Eigen Equation

- Starting with

$$\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}}$$

- Multiply both sides by $\bar{\bar{A}}^{\dagger}$

$$\bar{\bar{A}}^{\dagger}\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{A}}^{\dagger}\bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}}$$

- Hermitian conjugate of both sides

$$\left(\bar{\bar{A}}^{\dagger}\bar{\bar{V}}\bar{\bar{A}}\right)^{\dagger} = \left(\bar{\bar{A}}^{\dagger}\bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}}\right)^{\dagger}$$

- Since $\bar{\bar{V}}^{\dagger} = \bar{\bar{V}}$ & $\bar{\bar{T}}^{\dagger} = \bar{\bar{T}}$ & $\bar{\bar{A}}^{\dagger} = \bar{\bar{A}}$

$$\bar{\bar{A}}^{\dagger}\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{A}}^{\dagger}\bar{\bar{T}}\bar{\bar{\lambda}}^{\dagger}\bar{\bar{A}}$$

Checking Properties of □

- From

$$\bar{\bar{A}}^{\dagger}\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{A}}^{\dagger}\bar{\bar{T}}\bar{\bar{\lambda}}^{\dagger}\bar{\bar{A}}$$

- Using gives

$$\bar{\bar{A}}^{\dagger}\bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}} = \bar{\bar{T}}\bar{\bar{\lambda}}^{\dagger}$$

- And using

$$\bar{\bar{\lambda}}\bar{\bar{T}} = \bar{\bar{T}}\bar{\bar{\lambda}}^{\dagger}$$

- Which implies $\bar{\bar{\lambda}}$ is diagonal

$$\bar{\bar{\lambda}}\bar{\bar{T}} = \bar{\bar{T}}\bar{\bar{\lambda}}^{\dagger} \rightarrow \bar{\bar{\lambda}}\bar{\bar{T}} = \bar{\bar{T}}\bar{\bar{\lambda}} \rightarrow \bar{\bar{\lambda}} = \bar{\bar{\lambda}}^{\dagger}$$

Giving **A**

- $\bar{\bar{\lambda}}$ is real

  → λ_{ij} is real for all i, j
- Looking again at

$$\bar{\bar{V}}\bar{\bar{A}} = \bar{\bar{\lambda}}\bar{\bar{T}}\bar{\bar{A}}$$

- We can choose that $\bar{\bar{A}}$ is real